Felipe Bergamin Boralli

# Sensible Differences: Current account sensitivity in a small open economy with heterogeneous agents

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An open economy model with heterogeneous agents that models the effects of inequality on the economy

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An open economy model with heterogeneous agents that models the effects of inequality on the economy

São Paulo, January/2018:

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São Paulo
2018

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# Abstract

This work applies the recent methodological advances of the models of general equilibrium with heterogeneous agents, modifies such models so they are applicable to open economies. It is the possible to find the equilibrium states of said economy given relevant parameters. Also we study the dynamic effects of shocks of this economy. In a second part we calibrate the model to mimic the Canadian economy and we reach several similar moments of this calibration using our model.

**Keywords**: open economy, heterogeneous agents, inequality

# Resumo

Esse trabalho usa a recente metodologia de modelos de equilibrío geral de agentes heterogêneos, porém modifica tais modelos para que sejam aplicáveis a uma economia aberta. Assim, é possivel calcular os estados de equilíbrio desta economia para seus parâmetros relevantes, e seus efeitos na desigualdade da população desta economia. Esses efeitos são estudados em equilíbiro e seus efeitos transientes. Depois é feita uma calibração dos parâmetros para a economia canadense, e feita uma simulação estocastica dos momentos desta economia com a economia canadense. Nesta comparação, alguns momentos ficam bastante parecidos.

**Palavras-chaves**: economia aberta, agentes heterogêneos, desigualdade

# Summary

# List of Figures

# List of Tables

# 1 Motivation

The models of representative agents are the foundation of most of the accepted macro theory that is used today by academics and policy makers. However, the aggregation of the representative agent when transferred to reality could mask significant differences between the participants, which would lead to a different path of the economy for a given group of participants.

As such, the correct modeling of heterogeneity could , in turn, enhance the explanatory power of such models as the effects of heterogeneous agents could in aggregate provide a different picture of the economy as a whole. Also, this effect could be more significant in the presence of incomplete markets, where the agents would be exposed to a uninsurable risk, as stated in (HEATHCOTE, 2005), where the incomplete markets amplify the effects of a negative shock compared to a benchmark representative agent model. There is a large corpus of literature in the recent years that focus on the causes and effects of inequality, however the theoretical modeling of these phenomena is very recent, in part due to difficulties in modeling the distribution of the agents.

When aggregated, most phenomena related to inequality disappear, including any moments of the distribution of the agents and the dispersion of income, wealth, consumption and other relevant variables are not measurable in this case. There is relevant works (GALOR; ZEIRA, 1993) that claim this very effects caused by distribution are relevant to the cross-country differences found in the adjustment process of the world economies to exogenous shocks. Also, if one want to rigorously model the dynamics of inequality, and its effects on the long term evolution of welfare, either directly on indirectly via elections and governments as implied in (BENABOU, 1996) this methods open this theoretical avenue, once before open only to empirical analysis on data.

The treatment and proper modeling of inequality could open a new channel of transmission of shocks while also affecting the existing channels, either by increasing or decreasing their power, lags, effects, etc.

Also, in incomplete markets, consumers tend to exhibit self insuring behavior, that sometimes manifests itself in the form of inefficient equilibria (precautionary savings, for instance), where the modeling of heterogeneity is needed to visualize which parts of the population are self insuring, as in (HUGGETT, 1993).

One attempt to solve this and model the effects of inequality, is to break the representative agent class of consumers in two or more types. This maneuver attenuates the problem, but does not solve it. Also it cannot model the intricacies of the evolution of a distribution. A more recent attempt is to model a continuum of differences between agents in a distribution, and to model the properties of the full distribution.

There is a crescent literature in the field of continuum heterogeneous agents, but to the best of my knowledge, there is no work done for small open economies. The effects of the heterogeneity of the agents for the small open economy would be investigated, which could provide a solution of some unexplained behavior of said economies. The main interest of this paper is to verify the effects of heterogeneity in the flow and stock of external assets of this economy.

# 2 Introduction

The introduction of the heterogeneous agents in the model brings the phenomenon of emergent behavior, when complex actions of the whole emerges from the simple actions of the agents (GATTI et al., 2008). This is verified in economics and in other fields of knowledge (EPSTEIN, 2006). The complex behavior is not predictable from the simple rules that govern the agents, but from the complex set of interactions that happen from their relationships. We will see in this paper examples of this behavior, when changes in parameters will cause large nonlinear changes in the distribution of consumers.

Models that account the distribution of the agents have first appeared for discrete time, given the seminal work of (RÍOS-RULL, 1995), (AIYAGARI, 1994) and (HUGGETT, 1993). As an example of application, the work of Hugget calculates the effects of precautionary savings from the agents due to the fact of incomplete markets for loans (barrier to the size of a loan) lead to a economy-wide interest rate that is 1% lower than that of the complete economy.

However the seminal works above are all solved in discrete time, a choice that while also greatly facilitates the solution also introduces noise in the calculated theoretical distribution in the form of discrete jumps/spikes in income and wealth over a single time step generating spikes in the distribution of relevant variables, as shown in Figure 1, where we would expect a smooth distribution of agents instead.

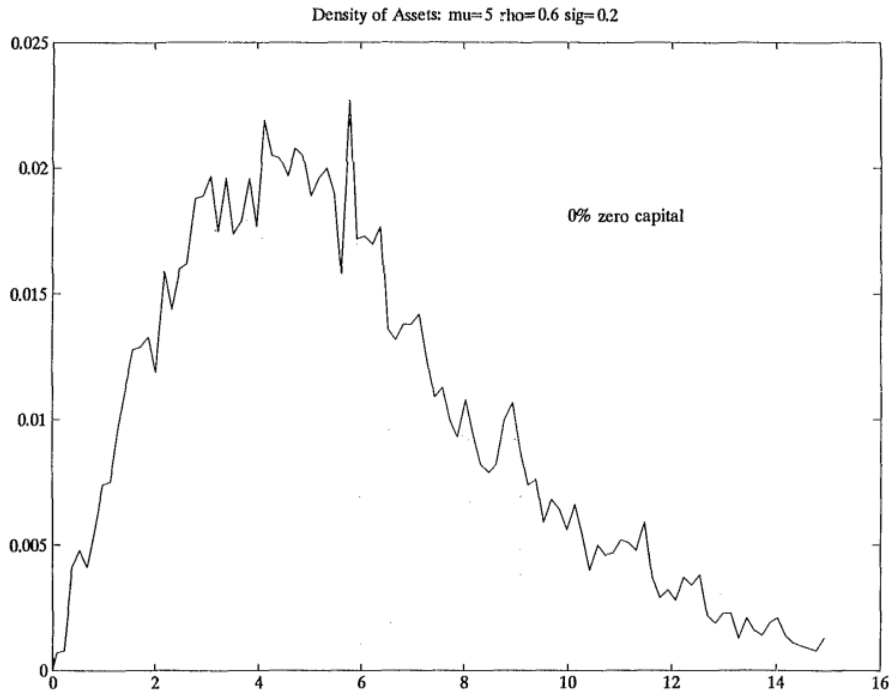


Figure 1 – Steady state theoretical wealth distribution of Aiyagari,1994:Fig5b. The distribution is not smooth due to the solution being done in discrete time.

Another effect of using discrete time for the solution is that any barrier on the state space of variables, like the lower limit for loans mentioned above must be enforced in a special step of the numeric solution. However we will see that this is not the case on continuous time models, where these barriers appear as a natural boundary condition of the problem.

Recently there has been new work that solves these models of heterogeneous on continuous time (ACHDOU et al., 2014) that is better suited for the type of studies we are proposing here. A model based on this approach will be used on this study.

# 3 Theoretical Model

Building upon work of Achdou, et al, Heterogeneous Agent Models in Continuous Time (ACHDOU et al., 2014), we simulate the economy of a small open country, to quantify the effect of the distribution of consumers income in the trade balance.

This version of model is a endowment open economy continuous time version of (HUGGETT, 1993), which is often referred as one of the seminal papers in the literature of heterogeneous agents.

The consumers are modeled as a continuum of income, and are subject to a uninsurable inferior limit on the liquid assets they possess. Shocks come as a change of this income. Given that the interest rate is exogenous in this economy, any surplus or deficit from the netting out of flows must come from abroad. Then, the effect of a change of the income distribution the external account would be derived.

## 3.1 Households

Households are the only agents of this model subject to heterogeneous effects, because we want to model distributional effects of the families assets on the economy.

Each household owns assets $a$ subject to a income of $z$ , and each one of them being on a continuous distribution of liquid assets $a$ and income $z$, which is a time varying endowment. The distribution of assets for the households is continuous given by $g(a, t)$. The distribution of income is a markov process with two states, one being higher income $z_j$, and one being lower income $z_{-j}$, so that $z_j > z_{-j}$. The transition probabilities from each state of the markov process are given by the Poisson distribution , conditional on the time already spent $(\Delta)$ on that state $P_j(\Delta) = e^{-\lambda_j \Delta}$.

Households want maximum utility in a infinite horizon:

$$\underset{\{c_t\}}{maxE} \int_0^{\infty} e^{-(\rho)t} u(c_t) dt \tag{3.1}$$

subject to:

$$\dot{a}_t = r_t^a a_t + z_t - c_t \tag{3.2}$$

$$z(t) = \begin{bmatrix} P_j(\Delta) & 1 - P_j(\Delta) \\ 1 - P_{-j}(\Delta) & P_{-j}(\Delta) \end{bmatrix} \begin{bmatrix} z_j \\ z_{-j} \end{bmatrix} \tag{3.3}$$

$$a_t \geq \underline{a} \; (\underline{a} < 0) \tag{3.4}$$

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma} \tag{3.5}$$

where c is consumption: Equation (3.2) is an accounting constraint on the liquid assets(cash accounts, government bonds), equation (3.3) is their exogenous income, given by a markov process with the transition matrix elements being poisson processes, and (3.4) a minima constraint on the account.Equation (3.5) is the standard CRRA utility function. Other utility types could be used, but this one is chosen for its tractability.

Income shocks happen randomly for each agent, and cannot be insured away due to incomplete markets (the lower bound). If a household with low assets faces a temporary reduction of income they will be forced to reduce consumption accordingly, in contrast with a standard complete market model where they would engage consumption smoothing via loans.

The lower limit of assets $\underline{\mathrm{a}}$ is the sole source of the incompleteness in markets. One could argue that different households have different debt limits (MAGRI, 2007), but for simplicity we assume that the limit is the same.

## 3.2 External Sector

The external sector of this model in a simplified way to introduce global asset markets, in which the households can participate, and a mechanism for the interest rate of said markets to affect the domestic economy.

Domestic households can give or take loans from each other or from the larger economy. Given this is a small open economy, the rate for these loans is fixed exogenously from the rest of the world. The amount that is not netted other domestic households results in a external account surplus/(debit when negative) $\dot{B}_t$, governed by the accounting identity, where $g(a,t)$ is the density of households in the distribution:

$$\dot{B}_t = \int_{\underline{\mathrm{a}}}^{\infty} g(a,t)\dot{a}_t da \tag{3.6}$$

The small open economy, as a whole, now has external assets $B_t$ (debts when negative), given in two ways by:

$$B_t = B_0 + \int_0^t \dot{B} dt = \int_{\underline{\mathrm{a}}}^{\infty} g(a,t) a_t da \tag{3.7}$$

There is also a no-Ponzi game condition enforced by the rest of the world where the small open economy cannot borrow in equilibrium more than the interest service, so that households cannot infinitely borrow not repaying their debts:

$$\lim_{t \to \infty} (B_t > \underline{\mathrm{B}} < 0) \tag{3.8}$$

This equation guarantees no explosive debt service.

# 4 Solving the Optimal Allocation Problem

In this section we will convert the equation of preferences and restrictions of the economy into a system of equations to be solved in the following section.

The decisions of the continuum of individuals will be governed by two equations: the Hamilton-Jacobi-Bellman (**HJB**) equation, which governs the optimal allocation between consumption or investment of each individual over time, and the Fokker-Plank equation (also called Kolmogorov forward, **KF**), which govern the evolution of the distribution of agents.

The **HJB** equation is the continuous time form of the Bellman equation commonly used in economics, and the derivation process for the two of them is similar. The complete derivation of the HJB is in the appendix. The HJB equation is this form models the actions of each of the agents, in response to aggregate effects of the economy, and the agent's own state variables. It governs the inter-temporal substitution and utility optimizing behavior of each agent.

The **KF** equation controls the movement of the distribution of the agents over time. The intuition behind it is that for each one tailed section $]-\infty, t]$, $\forall t$ of the distribution, the flow of agents must be conserved, since no agent appears or disappears. So any flow over that frontier must sum agents that changed distribution and the change in the mass of that section. The full derivation of the KF equation is in the appendix.

## 4.1 Reaching the Transient System of PDEs

We will first show the transient case and then modify it for the steady state case. The steady state system of PDEs is a particular case of the transient, more general case below. Each of the two states of the Markov chain will be modeled separately, a a condition imposed that on the transition and on the fact that they must sum up to one. The states are called by the $j$ and $-j$ modifiers below the variables, and both HJB and KF are functional equations, so we will not write both equations, as one is the mirrorer version of the other.

Most of the variables are function of either $a$ , $t$ or both, which were omitted for brevity. We define the optimal savings policy as $s_j = z_j + ra - c$ for brevity:

$$\rho v_j = \underset{c}{max} \left\{ u(c) + s_j \frac{\partial v_j}{\partial a} + \lambda_j (v_{-j} - v_j) + \frac{\partial v_j}{\partial t} \right\} \tag{4.1}$$

HJB

$$\frac{\partial g_j}{\partial t} = -\frac{\partial s_j g_j}{\partial a} - \lambda_j g_j + \lambda_2 g_{-j} \tag{4.2}$$

KF

$$1 = \int_{\underline{a}}^{\infty} g_j da + \int_{\underline{a}}^{\infty} g_{-j} da \tag{4.3}$$

$$B = \int_{\underline{a}}^{\infty} a g_j da + \int_{\underline{a}}^{\infty} a g_{-j} da \tag{4.4}$$

Boundary conditions so that the inferior barrier is respected. The saving $s$ at the barrier $\underline{a}$ must be greater than zero so that nobody crosses the barrier:

$$s_j(\underline{a}) = z_j + r\underline{a} - c_j(\underline{a}) \geq 0 \tag{4.5}$$

We present also a brief intuition behind the terms of the HJB and KF equations, the core of this paper: For the HJB equation, the first term is the utility, and the other terms are the terms that arrive from the laws of motion of the state variable $c$, or using the terms of dynamic programming (STOKEY, 2008), the first term is the return function, and the latter terms are the changes in value caused by infinitesimal changes in $a$,$\lambda$,$t$ respectively, related to Bellman's optimality principle.

As for the KF equation, the intuition is more straightforward, the left term (changes in density over time, for all points), must equal to flows over that point(partial derivative, the first term of the right), plus net flow of individuals that leave or enter that distribution in that point. This equation is valid for all the points of the distribution, and governs the flow of the distribution.

## 4.2 Reaching the system of PDEs in steady state

We define steady state the moment where no aggregate shocks appear, only individual shocks to income continue to happen. So for the steady state (**SS**) equations, we simply zero all time derivatives. As such, the distribution of agents is fixed, even tough there is a constant flow of agents moving between the two states of income, these flows net each other out, so that no aggregate quantities chage.

$$\rho v_j = \max_c \left\{ u(c) + s_j \frac{\partial v_j}{\partial a} + \lambda_j (v_{-j} - v_j) \right\} \tag{4.6}$$

HJB SS

$$0 = -\frac{\partial s_j g_j}{\partial a} - \lambda_j g_j + \lambda_2 g_{-j} \tag{4.7}$$

KF SS

$$1 = \int_{\underline{\text{a}}}^{\infty} g_j da + \int_{\underline{\text{a}}}^{\infty} g_{-j} da \tag{4.8}$$

$$B = \int_{\underline{\text{a}}}^{\infty} a g_j da + \int_{\underline{\text{a}}}^{\infty} a g_{-j} da \tag{4.9}$$

# 5 Numerical solution of the system of PDEs

Now that we have all the equations 'translated' from the optimal control (maximization) problem into the form of a system of partial differential equations that have no analytical solution, we will focus on the numerical strategy of their solution. The numerical solution will be used in the form of finite elements, where a grid is devised, and the derivatives are transformed into small differences.

## 5.1 Solving the system of PDEs using finite element analysis in steady state

We will use the standard way of solving a finite element problem: create an array over the entire range of values $a$ and approximate the HJB equation linearly in each of the values of the range. If the current value of the cell is lower, its value is nudged upwards a little, and vice-versa. This is done in several iterations until the variation of each cell is below a defined threshold, and the maximization of the HJB is considered complete. This method is called a viscosity solution of the HJB, and it is shown (CRANDALL, 1997) that is one approach to the numerical solutions of such equations that generates unique smooth solutions for HJB equations.

We first define the numerical approximate form of the derivatives of $v$ where $i$ is the index of the matrix of discrete $a$ values

$$\begin{cases} v_j'(a_i) \approx \frac{v_j(a_{i+1})-v_j(a_i)}{\Delta a}(forward) \\ v_j'(a_i) \approx \frac{v_j(a_{i-1})-v_j(a_i)}{\Delta a}(backwards) \\ v_j'(a_i) \approx \frac{v_j(a_{i-1})-v_j(a_{i+1})}{2\Delta a}(central) \end{cases} \tag{5.1}$$

This presents a problem of when to choose each of the approximations. This is treated in the literature for a class of problems which this one belong, and is called the **upwind scheme**, better explained in the literature, which we will give the general intuition here. As in the case of this problem, we generally expect a single maximum and a generally smooth distribution that increases, reaches that maximum, and decreases. If we fixed only one type of derivative, it would lead to accumulate errors to one side of the distribution as the iteration count progresses, due to numerical errors.

So, in the upwind scheme, we use the forward differentiation when $a'$ is increasing, ie. positive savings, backwards differentiation when $a'$ is decreasing, ie. negative savings, and central differentiation in the transition zone.

There is also the problem of numerical stability. We want this nudge to be in a way that will not be high enough that the system will oscillate over the equilibrium without reaching it. To accomplish this, there are two conditions: the Courant-Friedrichs-Lewy (CFL) condition, that specifies the size of the nudge, and the Upwind scheme that must be respected to reach a stable numeric solution.

So we present a simplified overview of the explicit method for the solution of the HJB:

1. Start with a guess for all the value function in a grid.
2. From this value function, calculate the numerical derivatives described above.
3. Compute the current consumption by doing the inverse of the CRRA value function, using the guesses above, using all derivatives above.
4. From the consumption, calculate savings of all the cells of the grid, using all derivatives, to decide where to use backwards, forwards or central derivatives, given the upwind scheme.
5. Now with the upwind scheme decided, compute the consumption of as above, but on the choice of derivatives for the upwind scheme.
6. Compute the both sides linearized version of the HJB equation. If the left hand side is lower than the right hand side, give a positive nudge on $v$ on that grid cell, and vice versa, for all cells. The size of the nudge should be small, and not zero out the differences of both sides, and should be below the CFL condition.
7. Iterate, using the current value grid as the next step initial guess. Stop when the size of all the nudges are below a threshold.

Once we reach the stop condition of the HJB equation, we now have the Value function of the Bellman problem, and the optimal savings and consumption policies associated with it at each point of the grid. Now we must linearize the KF equation and the unitary weight condition:

$$0 = -\frac{\partial s_j g_j}{\partial a} - \lambda_j g_j + \lambda_2 g_{-j} \tag{5.2}$$

$$1 = \int_{\underline{a}}^{\infty} g_j da + \int_{\underline{a}}^{\infty} g_j da \tag{5.3}$$

This discretizes in a grid with i and j as indexes :

$$0 = -g_j(i)\frac{\Delta s_j(i)}{\Delta a} - \lambda_j g_j(i) + \lambda_2 g_{-j}(i) \tag{5.4}$$

$$1 = \sum_{imin}^{imax} g_j(i)\Delta a + \sum_{imin}^{imax} g_{-j}(i)\Delta a \tag{5.5}$$

There is also a question backwards or forwards difference that must be used here as well, in the $\frac{\Delta s_j(i)}{\Delta a}$ term. However, as the KF is solved in a single step, and not iteratively, the authors of (ACHDOU et al., 2014) use for convenience one that mimics the same upwind scheme above, so the matrix used to inverse the KF solution is the similar to the one of the HJB matrix. We will use the same procedure.

## 5.2 Solving the system of PDEs using finite element analysis in transient

For our step response analysis, we always calculate the transient response from one steady state to another, with different parameters. This procedure greatly reduces the number of initial conditions that are needed to specify the initial state. If this was not the case, we would need to specify the starting distribution, with a large number of degrees of freedom. The simplified procedure for our transient solution is below:

1. Solve the steady state of the initial and terminal conditions.
2. Solve the transient HJB equation starting in the last period, ending in the first, using the Bellman principle of optimality. For each time period, the solution is very similar to the one above, except for the addition of transient terms.
3. With the HJB equation solved on the whole period, we now know the value function and the optimal savings policy for each agent across time.
4. In a similar fashion as before, we calculate the KF equation numerically for each timeframe, reaching the distribution of agents over time.

# 6 The steady state effects of the wealth inequality on the external credit: exploring the parameter space

In this section, calculate the steady state external credit for the economy, for various levels of the external exogenous interest rate. We use the parameters calibrated by (ACHDOU et al., 2014), but changing selected parameters at a time on chosen parameters at a time. The exploration of the parameter space is used to show relevant patterns of the steady state solutions and qualitative insights that could reveal hidden relations and correlations between our variables of interest, mainly the current account and external interest rate over the parameter space.

Table 1 – Baseline parameters

| Parameter | Value | Description |
|---|---|---|
| $\sigma$ | 2 | CRRA parameter for $u(c_t)$ |
| $\rho$ | 0.05 | Time discounting |
| $z_1$ | 0.1 | Lower income for the 2 state Markov income |
| $z_2$ | 0.2 | Higher income for the 2 state Markov income |
| $\lambda_1$ | 0.2 | Chance of switching from state 1 to state 2 |
| $\lambda_2$ | 0.3 | Chance of switching from state 2 to state 1 |
| $\underline{a}$ | -0.15 | Minimum constraint |

## 6.1 Income Inequality

In this series of changes, we calculate the equilibrium steady state of external credit $B$, for various levels of external interest rate, and varying differences between $z_1$ and $z_2$. We generate the variable $varIn$, short for variation of inequality:

Table 2 – Variation of income inequality

| $varIn$ | $z_1 = z_1^{base} - varIn$ | $z_2 = z_2^{base} + varIn$ |
|---|---|---|
| -0.050 | 0.150 | 0.150 |
| -0.025 | 0.125 | 0.175 |
| 0.000 | 0.100 | 0.200 |
| 0.025 | 0.075 | 0.225 |
| 0.050 | 0.050 | 0.250 |

When we calculate the net credit of this economy in steady state, over these values of $z$ for a range of external interest rates $r$. Overall, what we see in figure 2 is that for a given external interest rate, higher inequality of incomes leads to a higher net credit $B$, which is monotonicaly increasing for increases in $r$.

We see this observation of higher assets for higher inequality as a form of self-insuring precautionary savings of the higher income part of the population, so that they can sustain a higher consumption level whenever they 'fall' from the higher income state to the lower income state. This precautionary external savings of the small open economy explain one possible solution to the Lucas paradox described in (LUCAS, 1990). In this paradox, developing countries should not invest in developed countries, due to diminishing returns of scale in more mature economies, however this is not verified. This effect seem here, where the developing country's rich population oversave (in relation to complete markets) to self insure a future fall of income is a plausible explanation of this paradox. To the best of our knowledge investigations of this effect (as in (ALFARO; KALEMLI-OZCAN; VOLOSOVYCH, 2008)) have focused mainly on the investing side of the puzzle, and not on the savings side which is our case. While this chart in itself is not a solution to the puzzle, it is a starting avenue for further research.

We then fix the external credit rate at $r = 2\%$ and and show the distribution of the population in figures 3 and 4 where we see that a higher income inequality leads to a flatter, longer tailed distribution, and a larger mass on the lower limit $\underline{a}$, and more equal distribution leads to a narrower distribution.

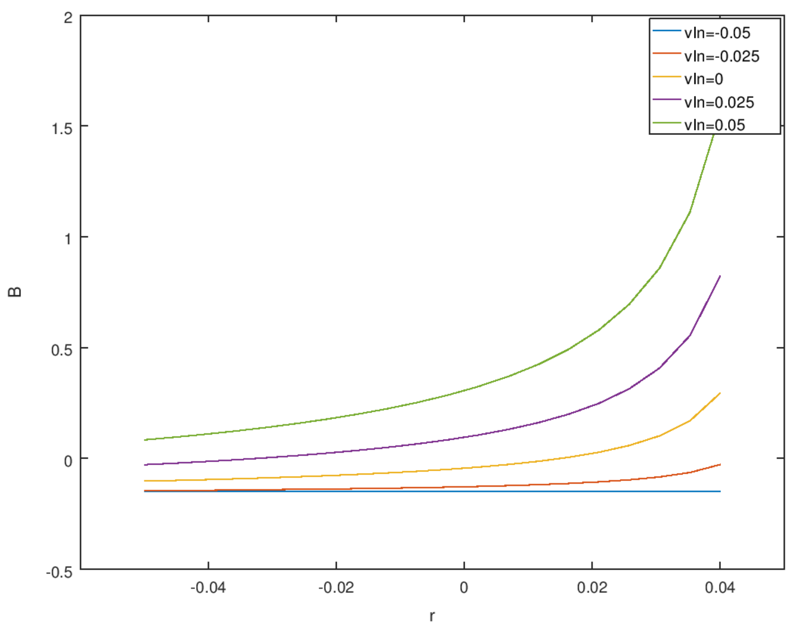


Figure 2 – Steady state curves of external credit for different income inequalities

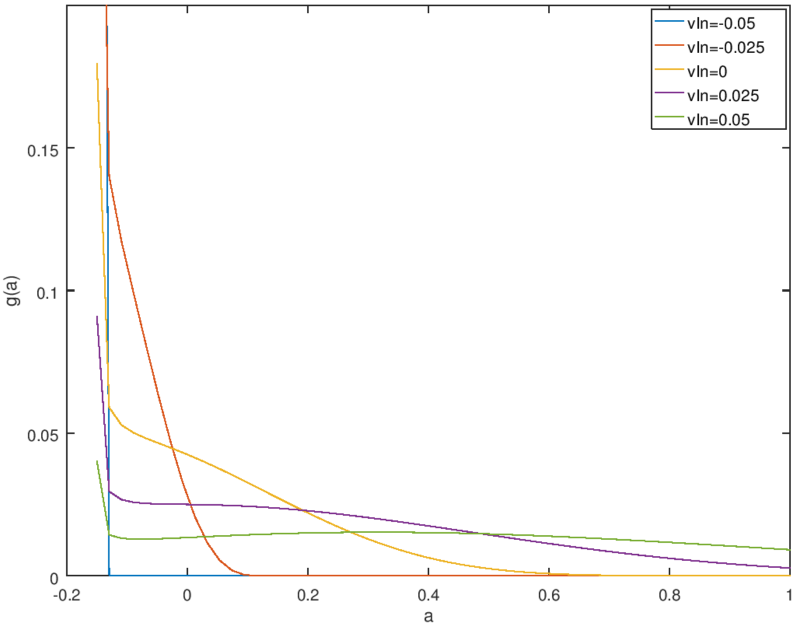


Figure 3 – Asset distributions for a fixed interest rate of 0.02 for variations of income inequality.

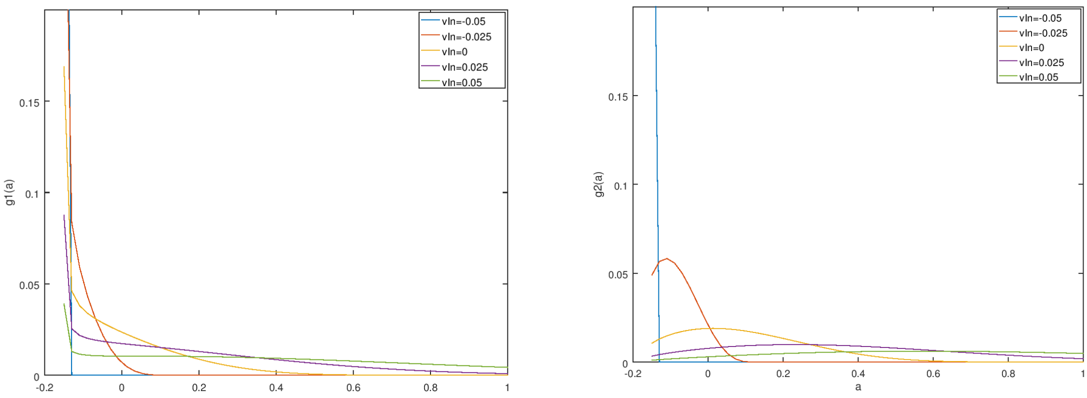


Figure 4 – Asset distributions for a fixed interest rate of 0.02 for variations of income inequality, with a breakdown per income state( $g_1$ is distribution the lower income population, $g_2$ is the distribution of the higher income).

## 6.2 Mobility

In this series of changes in parameters, we calculate the equilibrium steady state of of external Credit, for various levels of $\lambda_1$ and $\lambda_2$, which govern the flow of consumers between high and low states of income, called here as mobility. We generate the variable $vLamb$, short for variation of the lambda parameters:

Table 3 – Variation of income mobility

| $vLamb$ | $\lambda_1 = 10^{varLamb}\lambda_1^{base}$ | $\lambda_2 = 10^{varLamb}\lambda_2^{base}$ |
|---|---|---|
| -2 | 0.002 | 0.003 |
| -1 | 0.02 | 0.03 |
| 0 | 0.2 | 0.3 |
| 1 | 2 | 3 |
| 2 | 20 | 30 |

When we calculate the net credit of this economy in steady state, over these values of $\lambda_1, \lambda_2$ for a range of external interest rates $r$. Overall, what we see in figure 5 is that there is a complex relation of mobility and net external credit, given that $vLamb = -1$ that leads to the highest $B$, showing non monotonic behavior.

We then fix the external credit rate at $r = 2\%$ and and show the distribution of the population in figures 6 and 7 where we see a very complex shift in the shape of the curves from high to low mobility, and for the curve with max $vLamb = -1$ we see a bimodal shape of the distribution.

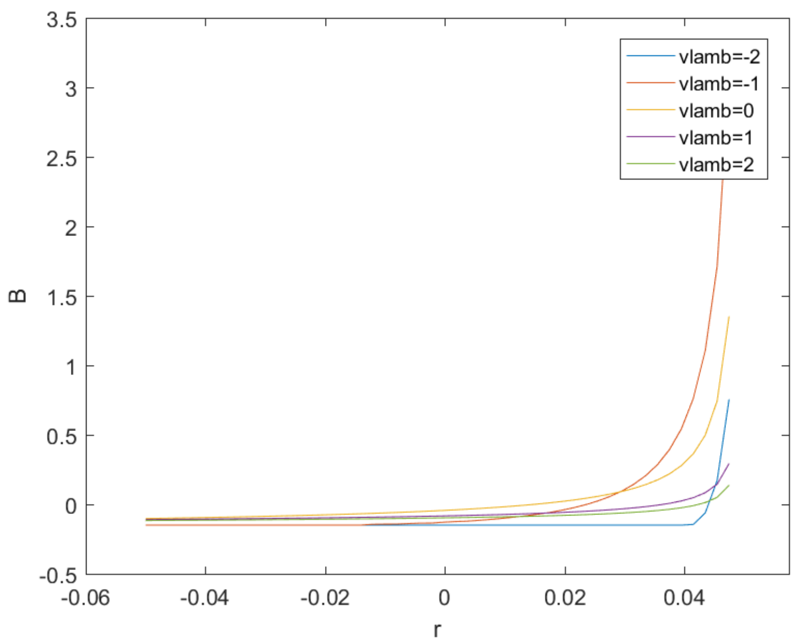


Figure 5 – Steady state curves of external credit for different external interest rates for variations in income mobility.

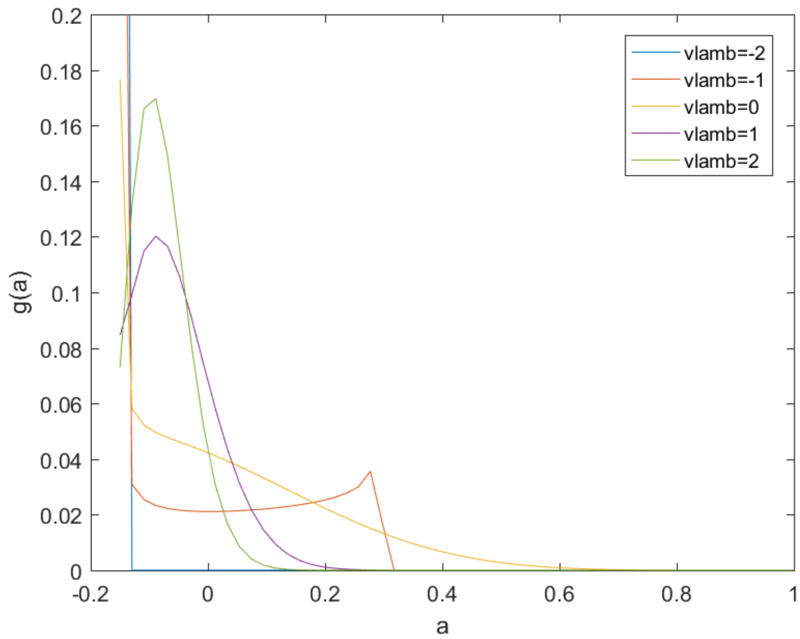


Figure 6 – Asset distributions for a fixed interest rate of 0.02. When the mobility is low, a two-crest, bimodal distribution appears.

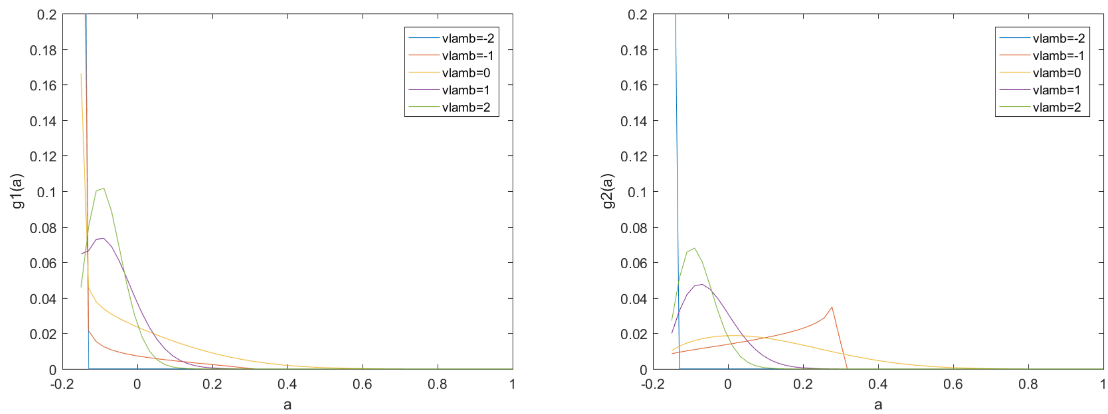


Figure 7 – Asset distributions for a fixed interest rate of 0.02. When the mobility is low, a two crest, bimodal distribution appears . $g_1$ is distribution the lower income population, $g_2$ is the distribution of the higher income.

## 6.3 Credit minima

In this series of changes in parameter values, we calculate the equilibrium steady state of of external Credit, for various levels of $\underline{a}$, in order to view the effects of the credit market in steady state distribution of the population. We generate the variable $vAmin$, short for variation of the minima constraint:

Table 4 – Variation of minima constraint

| $vAMin = \underline{a}$ |
|---|
| -4.0 |
| -3.0 |
| -2.0 |
| -1.0 |
| 0.0 |

When we calculate the net credit of this economy in steady state, over these values of $\underline{a}$ for a range of external interest rates $r$. Overall, what we see in figure 8 is there is a monotonic relationship between lower credit constraint and lower external credit assets $B$.

We then fix the external credit rate at $r = 2\%$ and and show the distribution of the population in figures 9 and 10. What we see is that as the minimum constraint is gradually reduced, the distribution is less and less bound by its value, and gets shaped into a smoother curve, but at a lower level of assets.

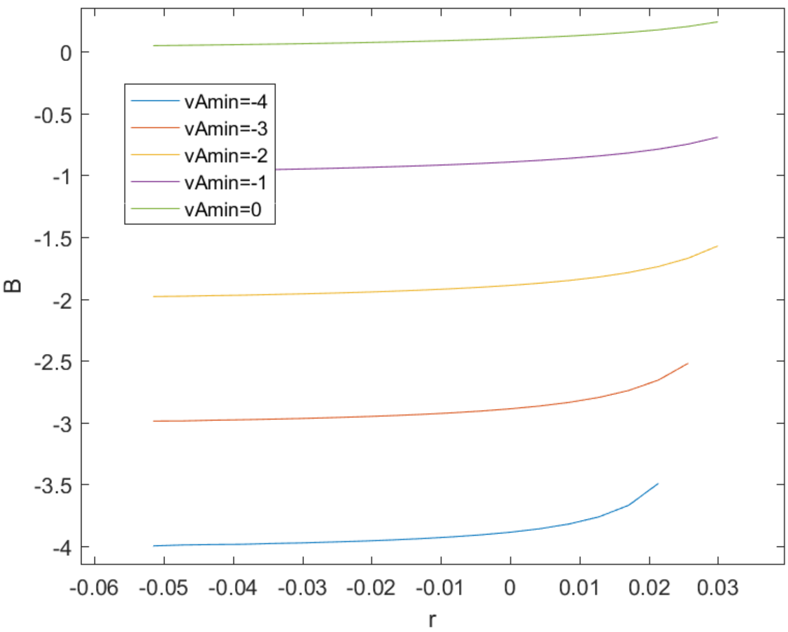


Figure 8 – Steady state curves of external credit for different credit constraints.

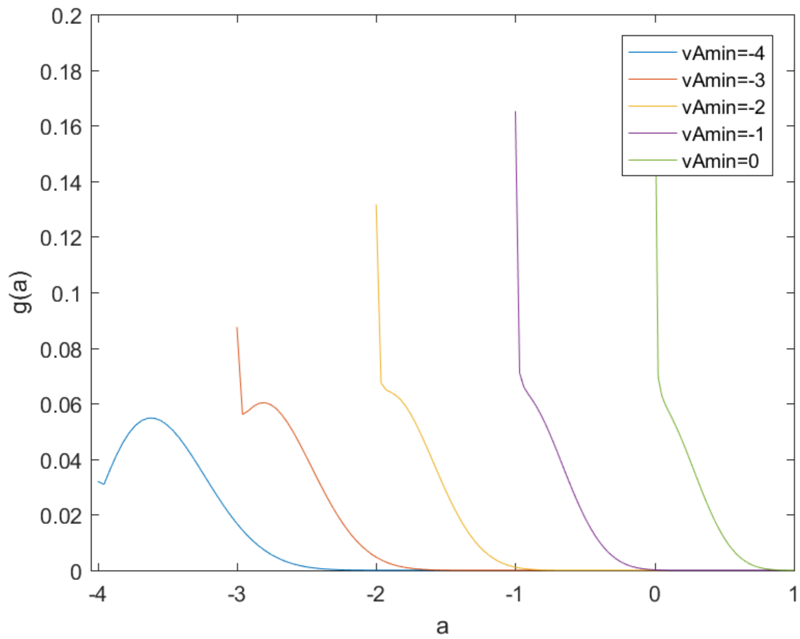


Figure 9 – Asset distributions for a fixed interest rate of 0.02 for different credit constraints.

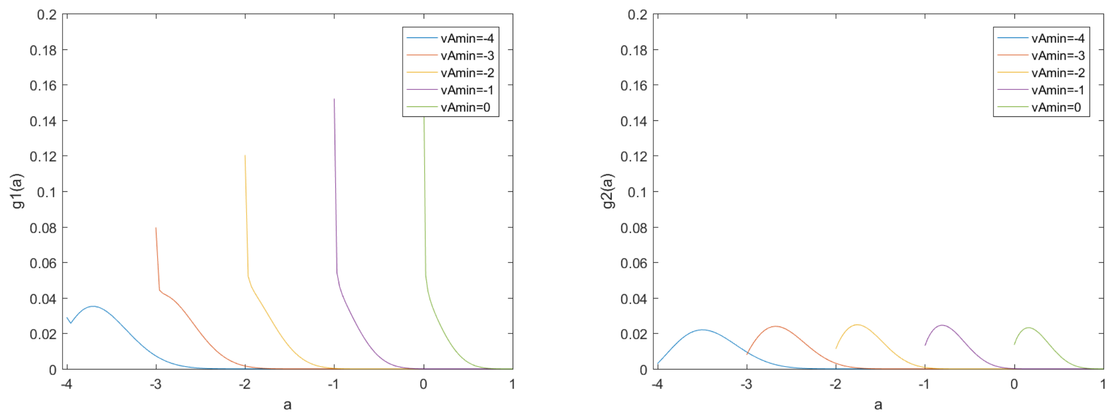


Figure 10 – Asset distributions for a fixed interest rate of 0.02, breakdown by income state. $g_1$ is distribution the lower income population, $g_2$ is the distribution of the higher income.

# 7 Analysis of a Step Impulse for a Heterogeneous Economy

We now explore on the solution over time of shocks in parameters of the model. In this analysis we can model the whole distribution over time, and its effects on the aggregate properties of the economy, so we can measure properties and moments of the distribution over time are not measurable over time for a representative agent model economy.

As of now, the model is linear, so it does not have a unique response to stimuli, but instead it depends on the region the shock is given. In all these step responses, we will model the transience of a permanent change from one steady state to another, given that the starting point is heavily dimensional, we see this as a simplifying assumption. Otherwise we would need to give starting values for all the distribution.

## 7.1 A income mobility shock

In this shock, we set $\lambda_1 = \lambda_2 = 0.5$ in the terminal state, keeping all other conditions constant, a a fixed interest rate of 0.03. We see in figure 11 that while the high income state see a monotonic decrease in assets over time, the low income consumers increase savings first, and after decrease then, resulting in a hump shaped effect for the whole population for savings over time. In equilibrium, the country's assets decrease from the starting point, after a period of higher assets. In figure 12, we see the complex transition of the distributions over time. In general, there is a shift to the left(lower savings) but this shift is first noticed in the higher income state and then on the lower income state. The tails of the distributions shirking over time, and a greater mass of consumers occupies the center of the distribution.

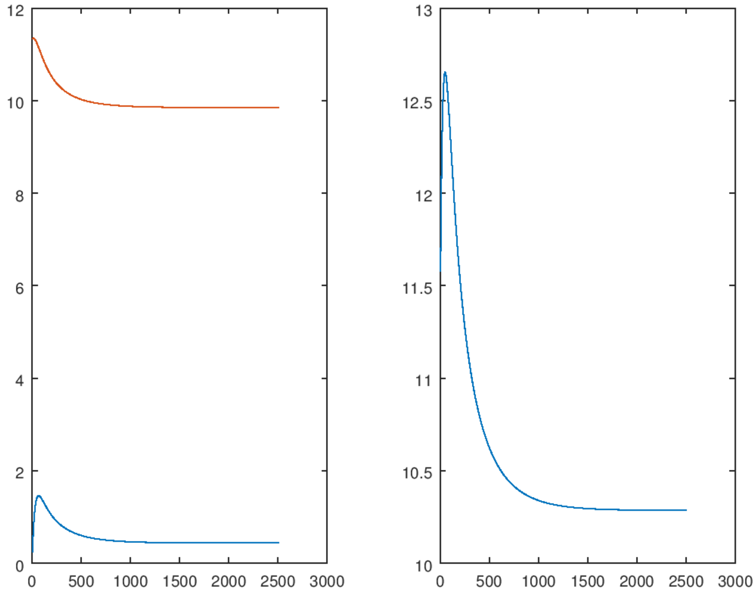


Figure 11 – Assets over time for a income mobility shock. On the left, the assets for each income state; on the right, the assets for the whole population.

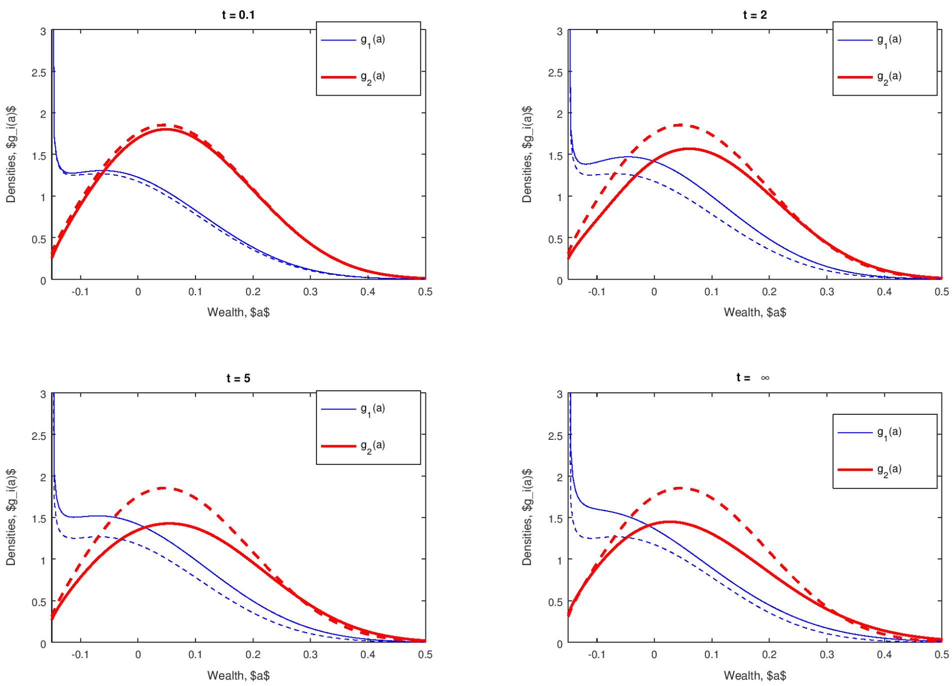


Figure 12 – Distribution over time for an income mobility shock. The dotted lines represent the initial state, drawn as a comparison. The solid lines are the distribution in the time tag.

## 7.2 Interest rate shock

In this shock we set the initial interest rate at 0.03 before the shock and 0.025 after the shock, and solve the model. All other parameters are set in their default values. In figure 13 we see that the assets of both income states decrease over time monotonically, in contrast to the mobility shocks seen before. The evolution of the distribution is mostly a shift to the left, as seen in 14.

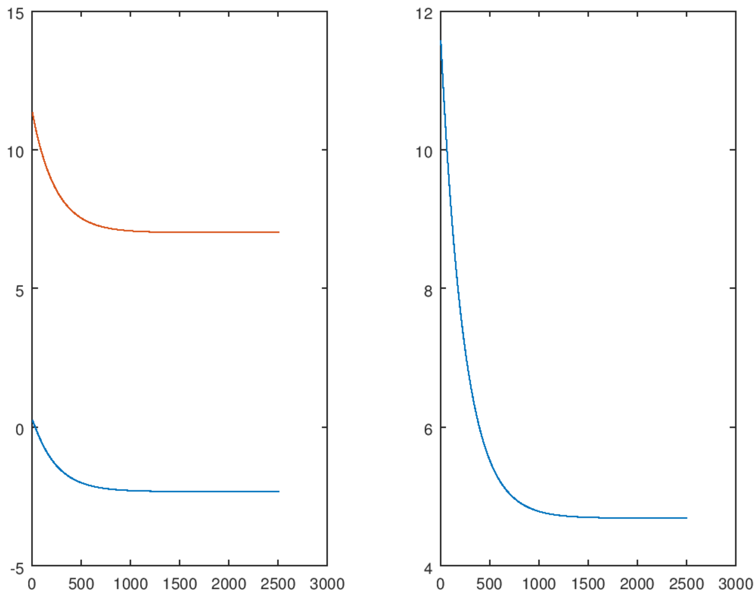


Figure 13 – Assets over time for an interest rate shock. On the left, the assets for each income state. On the right, the assets for the entire population

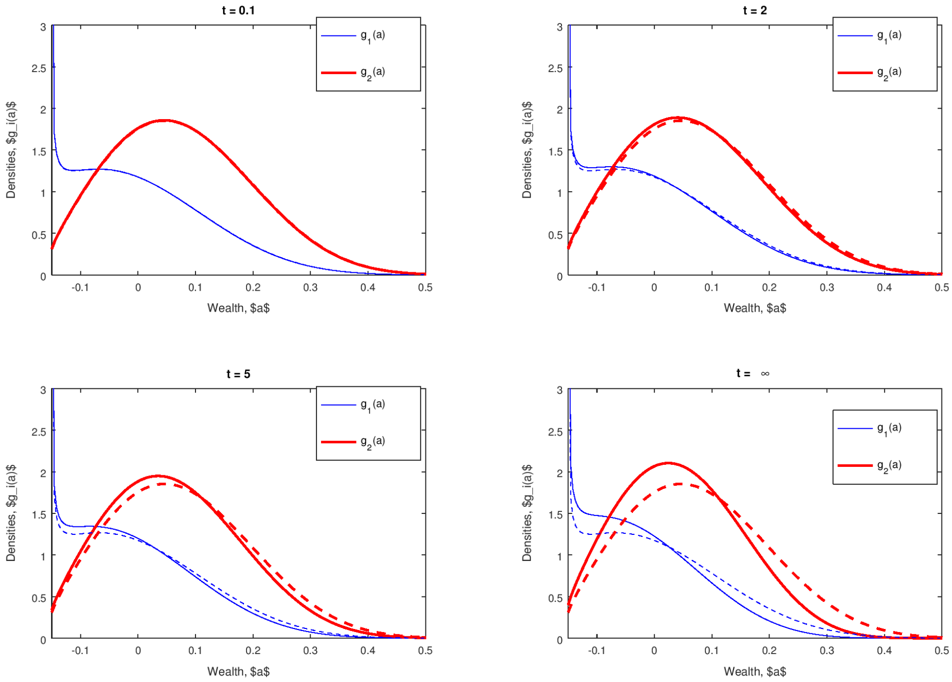


Figure 14 – Distribution over time for a Interest rate shock. The dotted lines represent the initial state, and are drawn as a comparison. The solid lines are the distribution in the time tag. This is the transition of one steady state to another (a step response), not a impulse response.

# 8 Stochastic simulation, and moments comparison

Given this field of heterogeneous agents in continuous time is relatively new, and as the time of this writing, no procedure for estimation given real data was devised, a favorable approach is the calibration of moments, which was also used in the seminal literature of representative agent models as in (KYDLAND; PRESCOTT, 1990) and (BAXTER; KING et al., 1991), when estimation of these models were also not possible.

Also, we focus on bringing information inequality. Given the richness of the distribution we will model calibrate a distributional moment: the Gini index, a well known measure of inequality, that goes from 0 for perfect equality to 1 for perfect inequality. Shocks to the economy can then be measured in terms of its effects in inequality.

We add three shocks to the economy, as small variations on the parameters $\lambda_1$,$r$ and $z_1$, to simulate the variance of unemployment, external interest rate, and salaries. We calibrate these parameters and the shocks to them to simulate the behavior of the Canadian economy. We choose to use Canada as our modelling coutry because it has large amounts of statistical data available and because it is also used by (MENDOZA, 1991), which we base this section of the paper. Therefore, we use a new set of parameters for this exercise, calibrated for the Canadian economy.

## 8.1 Database and data treatment

We use the following data series for the Canadian economy:

Table 5 – Data sources and treatment

| Variable | Source | Treatment |
|---|---|---|
| Consumption per capita | IMF | Log and HP filter |
| Wages (Employee compensation) | OECD | Divided per population,Log HP filter |
| External rate: Federal funds rate | FED | as is |
| Gini index (0-1 based) | World Bank | as is |
| Unemployment rate | OECD | HP filter |

Given that our model is very simplistic in its foundations, we use the two states of income to simulate employment and unemployment, and the frequency of transition between them to simulate the dynamics of the Canadian labor market. By this, we are able to bring real data into a very simple model. Please note that these parameters below are different from those used before.

Table 6 – Calibrated parameters for Canadian economy

| Parameter | Value | Description |
|---|---|---|
| $\sigma$ | 2 | CRRA parameter for $u(c_t)$ |
| $\rho$ | 0.05 | Intertemporal substitution |
| $z_1$ | 0.1 | Lower income for the 2 state Markov income : Unemployment bene |
| $z_2$ | 1 | Higher income for the 2 state Markov income : Employed |
| $\lambda_1$ | 0.5 | Average of unemployment is two quarters |
| $\lambda_2$ | $\lambda_1.(1-0.93)/0.93$ | Steady state unemployment rate of 7% = 1-93% |
| $\underline{a}$ | -2 | Arbitrary borrowing constraint of two salaries |

We calibrate three external shocks that will simulate the model: unemployment, wages and external interest rates. Given that this model has no output sector and no productivity which is often used as a source of shocks in these models we instead chose to model shocks to the endowment income as to mimic shocks coming from the output sector. The values for the parameters of the shocks were based on real data.

Table 7 – Calibrated shocks for Canadian economy

| Parameter | Value | Description |
|---|---|---|
| $\sigma_u$ | 0.36 | Standard deviation of the unemployment shock |
| $\rho_u$ | 0.9 | Autocorrelation of the unemployment shock |
| $\sigma_w$ | 1.2 | Standard deviation of the wage shock |
| $\rho_w$ | 0.8 | Autocorrelation of the wage shock |
| $\sigma_r$ | 0.9 | Standard deviation of the external interest rate shock |
| $\rho_r$ | 0.83 | Autocorrelation of the external interest rate shock |

We calibrate selected moments and simulate the system for a given period of time, a la (MENDOZA, 1991):[Table 1] for our economy, which is presented below:

Table 8 – Statistical Moments of the Calibrated Model for the Canadian Economy

| Variable | Correlation with Income | Correlation with Consumption | Standard Deviation |
|---|---|---|---|
| Income | 1.00 | 0.82 | 2.6% |
| Unemployment | -1.00 | -0.82 | 2.9% |
| Consumption | 0.82 | 1.00 | 6.0% |
| Savings measured in number of salaries | 0.08 | 0.63 | 185% |
| Current Account as % of GDP | -0.28 | -0.78 | 2% |
| Gini index | -0.31 | -0.52 | 0.02 |
| Average savings of the employed per person | 0.09 | 0.61 | 199% |
| Average savings of the unemployed per person | 0.05 | 0.64 | 185% |

Table 9 – Statistical Moments of the Real Canadian Economy

| Variable | Correlation with Income | Correlation with Consumption | Standard Deviation |
|---|---|---|---|
| Income | 1.00 | 0.78 | 1.54% |
| Unemployment | -0.83 | -0.72 | 0.59% |
| Consumption | 0.78 | 1.00 | 0.92% |
| Savings measured in number of salaries | N/F | N/F | NF/% |
| Current Account | 0.06 | 0.02 | 3.9% |
| Gini index | 0.31 | 0.26 | 0.01 |
| Average savings of the employed per person | N/F | N/F | N/F% |
| Average savings of the unemployed per person | N/F | N/F | N/F% |

Comparing the data of Table 8 and 9, we see that while some signs and overall magnitudes of the moments match, there are some severe discrepancies, mainly the one in the standard deviation of the savings, that could not be found in real data, but has a very high value of about 200% of standard deviation.

The Gini index, our measure of inequality, has the opposite sign of correlation on income and consumption compared to real data. While in our model a increase in income decreases inequality, in the real economy, it is increased.

# 9 Conclusion

This work had the objective of presenting the recent literature on heterogeneous agents in continuous time and to propose a new application of the methods in the analysis of a small open economy.

We derived and deduced the equilibrium and dynamic equations of this economy given its participants, showcasing the instrumental variables of this field, mainly the Hamilton-Jacobi-Bellman equation and the Kolmogorov-forward equation, their purpose in the model.

We also presented an overview of the decisions made when setting the partial differential equations of the model in a setup for numerical calculus, the partial differences and choices that must be made in setting up the problem.

The parameter space was searched, and several interesting properties emerged of this search. The population exhibits various different distributions conditional on these parameters, and the transitions of a shock in the parameters. We calculated step responses from one equilibrium condition to another. We see this parameter search as the most important contribution of this paper, as it showcases a new type of chart not previously (to the best of our knowledge) seem in the literature of heterogeneous agents in continuous time, seen in figures 2, 5 and 8. We uncovered a new plausible explanation to the Lucas paradox which, while incomplete, serves as an starting point for further research in the subject.

At last, we calibrated the model a la (MENDOZA, 1991) and came up to several similar moments, of the distribution, but in addition to the cited work, we also could model unemployment and the Gini index, a distributional property that is not measurable using representative agents models. However, in our model an increase of consumption or income decreases the Gini index, in the real data it increases following such event, showing that our model still has not uncovered some lingering relations of the real economy.

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# ANEXO A – The Hamilton–Jacobi–Bellman equation

## A.1 Warming up: the neoclassical deterministic case

We have the neoclassical optimization problem of the consumer in continuous time, and we want to solve for the optimal allocation path. We will use the 'Hamiltonian recipe', that is common for this type of problems.

$$\max_{c} \int_0^\infty e^{-\rho t} u(c(t)) dt \tag{A.1}$$

$$s.t\ \dot{a} = ra + w - c \tag{A.2}$$

The Hamiltonian of this problem:

$$\widehat{H}(a, c, \mu) = u(c) + \mu(ra + w - c) \tag{A.3}$$

And the conditions on it:

$$\frac{\partial \widehat{H}}{\partial c} = 0 \rightarrow \frac{du(c)}{dc} = \mu \tag{A.4}$$

$$\frac{\partial \widehat{H}}{\partial a} = u'(c) = \rho\mu(t) - \mu'(t) \rightarrow \frac{\dot{\mu}}{\mu} = -(r - \rho) \tag{A.5}$$

Now we derive the first condition on time, and substitute on the second:

$$u''(c)\dot{c} = \dot{\mu} \tag{A.6}$$

$$\frac{u''(c)\dot{c}}{u'(c)} = -(r - \rho) \tag{A.7}$$

To proceed, we need a form for the consumption utility, lets suppose $u(c) = log(c)$, and substitute in the previous equation, we reach the optimal decision of consumption:

$$\frac{\dot{c}}{c} = -(r - \rho) \tag{A.8}$$

## A.2 The HJB equation in the case of a Markov process

We will deduce the HJB equation from the problem of maximization of the household, subject to the effect of a two state markov. We will start with a discrete time formulation, and convert it to a continuous in the middle of the process:

$$V(a_t) = \sum_t^{\infty} \beta^t u(c_t) \tag{A.9}$$

subject to:

$$a_{t+1} = z_t + ra_t - c_t a_t > \underline{a} Markov = \begin{cases} State1 : P^{s1}_{change}(\Delta) = e^{\Delta\lambda_1} & P_{stay}(\Delta) = 1 - P^{s1}_{change}(\Delta) \\ State2 : P^{s2}_{change}(\Delta) = e^{\Delta\lambda_2} & P_{stay}(\Delta) = 1 - P^{s2}_{change}(\Delta) \end{cases} \tag{A.10}$$

the last being the transition matrix from the markov transition matrix. Using the Bellman optimality principle, which states: *"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."*, we isolate the first time period of the policy from the rest, taking into consideration the chance of changing states

$$V_1(a_t) = max\left[u(c_t) + \beta\left(P^{s2}_{change}V_1(a_{t+1}) + (1 - P^{s2}_{change})V_2(a_{t+1})\right)\right] \tag{A.11}$$

$$V_2(a_t) = max\left[u(c_t) + \beta\left(P^{s1}_{change}V_2(a_{t+1}) + (1 - P^{s1}_{change})V_1(a_{t+1})\right)\right] \tag{A.12}$$

and now taking $\Delta = t - (t-1) \to 0$ and using the linear approximation of the poisson process for small values. $\beta(\Delta) = e^{-\rho\Delta} \approx 1 - \rho\Delta$ and also $P^{s1}_{change}(\Delta) = e^{-\lambda\Delta} \approx 1 - \lambda\Delta$

$$V_1(a_t) = max\left[u(c_t)\Delta + (1 - \rho\Delta)\left((1 - \lambda_1\Delta)V_1(a_{t+1}) + \lambda_2 V_2(a_{t+1})\right)\right] \tag{A.13}$$

$$V_2(a_t) = max\left[u(c_t)\Delta + (1 - \rho\Delta)\left((1 - \lambda_2\Delta)V_2(a_{t+1}) + \lambda_1 V_1(a_{t+1})\right)\right] \tag{A.14}$$

Now subtracting $(1 - \rho\Delta)V_{(1|2)}$ from both sides and rearranging, we reach:

$$\Delta\rho V_1(a_t) = max\left[u(c_t)\Delta + (1 - \rho\Delta)(V_1(a_{t+\Delta}) - V_1(a_t) + \Delta\lambda_2(V_2(a_{t+\Delta}) - V_1(a_{t+\Delta})))\right] \tag{A.15}$$

$$\Delta\rho V_2(a_t) = max\left[u(c_t)\Delta + (1 - \rho\Delta)(V_2(a_{t+\Delta}) - V_2(a_t) + \Delta\lambda_2(V_1(a_{t+\Delta}) - V_2(a_{t+\Delta})))\right] \tag{A.16}$$

Now, using the limit of when $\Delta = t - (t-1) \to 0$ to get:

$$\frac{dV_1(a)}{da} = V_1'(a)(z_1 + ra_t - c) \tag{A.17}$$

$$\frac{dV_2(a)}{da} = V_2'(a)(z_2 + ra_t - c) \tag{A.18}$$

Finally reaching:

$$\rho v_1 = \max_c\left\{u(c) + \frac{\partial v_1}{\partial a}(z_1 + ra - c) + \lambda_1(v_2 - v_1) + \frac{\partial v_1}{\partial t}\right\} \tag{A.19}$$

$$\rho v_2 = \max_c\left\{u(c) + \frac{\partial v_2}{\partial a}(z_2 + ra - c) + \lambda_2(v_1 - v_2) + \frac{\partial v_2}{\partial t}\right\} \tag{A.20}$$

# ANEXO B – The Fokker-Plank equation

## B.1 For discrete states in a Markov

This demonstration is found at the online appendix of (ACHDOU et al., 2014), item B3 of the appendix. Not included here for brevity. We present the more general and intuitive demonstration for any diffusion below.

## B.2 For continuous states in a diffusion process

In the case of continuous states, we cannot use the approach used before, and we will not use any linear approximation. The strategy of this proof if to find the conservation of density over time. Given a density $g(a, z, t)$ of the assets and income that varies in time, so that:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(a, z, t)dadz = 1 \tag{B.1}$$

Let us also define a vector $\overrightarrow{x} = \begin{pmatrix} a \\ z \end{pmatrix}$, called the state vector, of a single, infinitesimal person. Now if we consider a section $S_0$ , with perimeter $p_0$ of the domain of the function $g$, and calculate the time variation of density on that section, that time variation must equal the flow of people though the frontier of the section, given that no people appear inside $S_0$, they only move over the $a - z$ field:

$$\frac{d}{dt}\int\int_{S_0} gdadz = -\int_{p_0} g\dot{\overrightarrow{x}} \cdot \overrightarrow{dS} \tag{B.2}$$

Putting the left time derivative inside, and using the Gauss theorem to convert the right term to a surface integral:

$$\int\int_{S_0} \frac{\partial}{\partial t} gdadz = -\int\int_{S_0} g\nabla \cdot \dot{\overrightarrow{x}}dadz \tag{B.3}$$

Given $S_0$ is arbitrary, we find the equation of variation of probability for any given point (compare the insides of the integrals):

$$\frac{\partial}{\partial t}g = -g\nabla \cdot \dot{\overrightarrow{x}} \tag{B.4}$$

$$\frac{\partial}{\partial t}g = -\frac{\partial}{\partial a}\dot{a}g - \frac{\partial}{\partial z}\dot{z}g \tag{B.5}$$

Please note the similarities of this result with those of the discrete case.