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Variational Bewley Preferences

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Abstract

This paper characterizes *variational Bewley preferences* over Anscombe and Aumann acts, a class of binary relations that may fail completeness or transitivity *vis a vis* independence. The main result gives an axiomatization of preference relations $\succsim$ represented as follows:

$$f \succsim g \Leftrightarrow \int u(f)\,dp + \eta(p) \geq \int u(g)\,dp \text{ for all } p \in \Delta,$$

where u is an affine utility index over a convex set X of consequences, $\eta : \Delta \to [0, \infty]$ is an ambiguity index, and Δ is the set of priors over the state space S. This representation has a natural interpretation as a *weighted unanimity rule*, with the function η reflecting the weight given to a prior and higher values of η corresponding to priors given less weight. Bewley's incomplete preferences can be identified precisely with the addition of transitivity or independence, and a prior receives weight either 0 if plausible or ∞ when discarded. Also, by adding only completeness, we recover subjective expected utility, *i.e.*, the lack of transitivity implies incompleteness. Finally, we find a strong connection of our model with the class of variational preferences.

Keywords: Ambiguity, Knightian uncertainty, incomplete preferences, intransitivity, variational preferences. *Journal of Economic Literature* Classification Number: D81.

1 Introduction

In many economic choice situations, incomplete preferences arise as a compelling possibility when the decision maker (DM) may find themselves unable to arrive

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at a ranking for certain pairs of alternatives. The context of choices under uncertainty is subject to this phenomenon when a precise probability measure over events cannot be supported by the available information. The Knightian decision theory, as proposed by Bewley (2002), characterizes DMs with incomplete preferences due to the multiple priors nature of beliefs, which is consistent with Aumann's (1962) complaint about the inaccurate description of actual behavior implied by the completeness axiom.

The Knightian decision theory describes a DM that holds a set of beliefs C and follows a *unanimity rule* given by a prior-by-prior dominance: an act f is at least as desirable as g if, and only if, the expected utility of f is not less than the expected utility of g, for all priors in C.[1] More probabilities there are characterizing a DM, the less susceptible this DM should be to a comparison with alternatives, and *vice versa*. Furthermore, when the decision maker cannot compare acts f and g, and some of these acts are assumed to be the starting point (the *status quo*), then the decision maker will keep the *status quo* as the default choice.

A natural interpretation for Bewley's preferences considers a DM as coping with many opinions revealed by experts or specialists.[2] The set of multiple priors C captures the universe of all opinions defended by some Bayesian expert, and one act is weakly preferred over another if its associated expected utility exceeds or equals that of the alternative according to every expert's opinion in the set. An important feature of this unanimity rule is that every opinion is treated with the same weight. In view of this, it is natural to wonder how restrictive is such uniform weighting? Let us consider a policy maker that wants to take a decision on a given subject, with the support of some leading economists. Since experts usually have different priors on scenarios, the policy maker likely faces the problem of a disagreement on the ranking of certain pairs of acts. For instance, we may think of the chairperson of the Fed asking economists for advice regarding monetary policy. Decisions made by the monetary authority are subjective regarding future states of the economy (*e.g.,* macro fundamentals). Assume that between two available acts f and g some economists (group A) claim that the net expected utility $E\left[u\left(f\right)-u\left(g\right)\right]$ is positive and high (a substantial gain), while other economists (group B) disagree and claim that the corresponding net expected utility is negative and close to zero (a small loss). Following Bewley's unanimity rule, the DM should not switch from g to f. We are concerned here with this restrictive feature of the unanimity rule, and the purpose of this paper is to present a more flexible dominance criterion, in the sense that a DM might take into account different pieces of advice with different weights.

[1] We follow the Ghirardato, Maccheroni and Marinacci (GMM; 2004)'s version of the Bewley's model. Ghirardato, Maccheroni, Marinacci and Siniscalchi (GMMS, 2003) provide a derivation of Bewley model in the purely subjective probability framework *a la* Savage. Faro (2013) axiomatized Cobb-Douglas incomplete preferences *a la* Bewley.

[2] Here, we assume that the DM's preference over consequences determines the attitude towards risk. Furthermore, what matters from the group of *Bayesian* experts is their opinion about the relative likelihood of each scenario.

Accordingly, our main goal in this paper is to provide a behavioral foundation for a model of decision making in which the DM's confidence can vary across different opinions or priors.[3] In the discussion about Fed decisions, the chairperson might attach more importance to some advice than other emerging from the whole group of experts. While the unanimity rule described by Bewley's model requires equal weight among all plausible priors, we aim to propose and axiomatize a model of choices under uncertainty with a weighted unanimity rule. Furthermore, one of the purposes of this paper is to characterize each weight given to a prior as a threshold of acceptable loss in terms of the net expected utility. Intuitively, if the chairperson ignores a given expert's negative advice about some course of action, up to a certain amount of loss, then such a payoff level constitutes the degree of importance of this expert.

The fundamental aspect for characterizing a weighted unanimity rule is its behavioral foundation. Clearly, it means that we must describe a list of behavioral properties weaker than those related to Bewley's model. Again in the Fed decision context, consider three options - namely f, g, and h - in which any conceivable ranking between them can emerge from the group of experts. From the perspective of Bewley's unanimity rule, the Fed should not move in any direction. However, the Chairperson might take the advice of some experts more seriously than others. First, imagine that the Chairperson classifies advisers in two groups: one contains the most relevant experts with full relevance, while the other group describe less relevant advisers. Also, assume that there is a complete agreement between those more important experts that both f is *much* better than g and g is *much* better than h. At the same time, there are contrary opinions saying instead that g is *weakly* better than f and that h is *weakly* better than g. But, such opinions are held by experts that the DM deems less important, and given the weak dominance revealed by such less reliable advisers, the DM follows the guidance given by the group of more important experts by revealing that $f \succ g$ and $g \succ h$. On the other hand, even under the natural consequence of the assumption that all top experts must agree that f is better than h, it is not immediate that the opinion saying that h is *weakly* better than f must emerge from less relevant advisers. Indeed, it could be the case that the DM cannot compare f and h because some less reliable expert *strongly* recommends h against f. To summarize, this reasoning concludes that transitivity can be violated because chains of weak negative recommendations can lead to strong negative advice.

Another possibility is a cycle with $f \succ g$, $g \succ h$, and $h \succ f$. Why might this happen? Assume a consensus based on the 'most important advisers' that f, g, and h are indifferent. On the other hand, suppose that any non full reliable expert agree that f is not much worse than g, g is not much worse than h, and that h is not much worse than f, which supports the weak rankings $f \succsim g$, $g \succsim h$,

[3] Some representations of complete preferences have captured a similar idea of different degrees of confidence over priors, *e.g.*, Maccheroni, Marinacci and Rustichini (2006) and Chateauenuf and Faro (2009). In another framework, Nau (1992) proposed a 'quasi-Bayesian' model in which beliefs are represented by lower and upper probabilities associated to confidence weights.

and $h \succsim f$. Finally, suppose that from three non negligible advisers we have the following opinions: g is much worse than f, h is much worse than g, and f is much worse than h. Under such conditions the DM might reveal the pattern of behavior $f \succ g$, $g \succ h$, and $h \succ f$.[4] This cycle is a consequence of conflictive opinions revealed by experts with low relative importance in a context where top ones are indifferent between them. If we accept the money pump argument usually used against cycles, we might interpret this fact as the price of listening people with less weight when the most important ones consider the involved acts as the same.

In terms of representation, we model the relative importance of each prior (or Bayesian expert) by an ambiguity index η over the set of priors Δ with values in $[0, \infty]$, where $\eta(p) \leq \eta(q)$ means that the model p is *more plausible or important* than the model q, and $\eta(p) = \infty$ means that the model p is discarded. The ambiguity index η follows the intuition of Savage (1954), pp. 57–58, saying that 'there seems to be some probability relations about which we feel relatively "sure" as compared with others.' The set of priors $\{p : \eta(p) = 0\}$ captures the relatively 'most sure' probabilities while other less plausible probabilities are related to a positive ambiguity index described by η.

The Main Representation Result

Aiming to get a model that captures the ideas discussed above, we characterize preference relations that may fail completeness or transitivity *vis a vis* independence. We consider a framework *a la* Anscombe and Aumann (1963), and our main result gives an axiomatization of preference relations $\succsim$, called *variational Bewley preferences*, represented by an affine utility function u over consequences and an ambiguity index η, where for any pair of acts f and g

$$f \succsim g \Leftrightarrow \int u(f)\, dp + \eta(p) \geq \int u(g)\, dp \text{ for all prior } p.$$

This representation has a natural interpretation as a weighted unanimity rule, with the function η reflecting the weight given to a prior and higher values of η corresponding to priors given less weight. Indeed, we show that the axiomatic foundation of variational Bewley preferences rests on a simple set of axioms that generalizes Bewley incomplete preferences model as proposed by GMM (2004). Being more precise, we do not impose transitivity and we use a different condition when compared to the classical independence axiom. We also provide a comparative notion for variational Bewley preferences.

Let us come back to the interpretation in which the set of multiple priors C denotes all possible opinions revealed by a group of relevant experts. In Bewley's model, all experts are Bayesian agents and the decision maker prefers an act f to an act g if, and only if, all experts agree with this ranking. In our model, the decision maker can associate different weights to different experts. In a sense, there is a ranking for experts captured by the ambiguity index η. Actually,

[4] See Example 13 for a numerical illustration of this possibility.

our decision maker finds that an act f is at least as desirable as g if, and only if, given any opinion represented by a prior p, the choice of f against g does not generate an expected utility loss greater than $\eta(p)$. One of the properties of η says that $\eta(p) = 0$ has at least one solution $p^* \in \Delta$. That is to say, for some opinion p^*, the decision maker does not accept any level of loss and any prior with this property identifies a fully plausible opinion. On the other hand, an opinion described by p with $\eta(p) \in (0, +\infty)$ represents a prior that is not completely ignored but our decision maker may find f better than g even if $\int (u(f) - u(g))\, dp$ is negative, provided that it is bounded below by $-\eta(p)$.

Connection with Variational Preferences

We obtain an interesting relation between variational Bewley preferences and the class of unbounded variational preferences introduced by Maccheroni, Marinacci and Rustichini (MMR, 2006). In the same style of the connection between Bewley's model and Maxmin preferences provided by Gilboa, Maccheroni, Marinacci and Schmeidler (GMMS, 2010), we show that given a variational Bewley preference $\succsim^*$, represented by (u, η), and a complete, monotone, and continuous preorder $\succsim^{**}$ such that jointly $(\succsim^*, \succsim^{**})$ satisfy *Weak Consistency* and *Default to Certainty*, then the preference $\succsim^{**}$ is a variational preference represented also, *mutatis mutandis*, by (u, η). Note that, under Weak Consistency and Default to Certainty, a variational representation of preferences can be derived without assuming the uncertainty aversion axiom of Schmeidler (1989) and without the weak certainty independence axiom of MMR (2006). We also obtain a result that shows how to retrieve a variational Bewley relation from a variational preference, which is inspired by the alternative way of writing our representation given by, for all acts f and g

$$f \succsim g \Leftrightarrow \min_{p \in \Delta} \left\{ \int (u(f) - u(g))\, dp + \eta(p) \right\} \geq 0.$$

Finally, some interesting subclass of variational Bewley preferences are briefly introduced when we consider the entropic and Gini ambiguity indexes.

Outline

The present paper is organized as follows. After introducing the setup in Section 2 and describing the axioms in Section 3, we present the main representation result in Section 4. In subsection 4.1 we show that Bewley's model can be identified precisely with the addition of transitivity or independence. In subsection 4.2 we show completeness recovers subjective expected utility. In subsection 4.3 we characterize, in terms of representation, when one variational Bewley preference displays more ambiguity than another. In subsection 4.4 we characterize the Bewley closure and the Bewley interior of any variational Bewley preference. In subsection 4.5 we present some examples that further illustrate the properties of our model. In Section 5 we provided a study with the connections between our model and the class of variational preferences, which also motivates the introduction of some interesting subclasses of variational Bewley preferences. The proofs and related material are collected in the Appendix.

2 Framework

Consider a set S of *states of nature (world)*, endowed with a σ-algebra Σ of subsets called *events*, and a non-empty set X of *consequences*. We denote by $\mathcal{F}$ the set of all the (simple) *acts*: finite-valued functions $f : S \to X$ which are Σ-measurable.[5] Moreover, we denote by $B_0(\Sigma)$ the set of all simple real-valued Σ-measurable functions $a : S \to \mathbb{R}$. The norm in $B_0(\Sigma)$ is given by $\|a\|_\infty = sup_{s\in S} |a(s)|$ (called *sup norm*) and $B(\Sigma)$ will denote the supnorm closure of $B_0(\Sigma)$. In another way, $B_0(\Sigma)$ is the vector space generated by the indicator functions of the elements of Σ, endowed with the supnorm (for more details, see Dunford and Schwartz (1958), section 5 of chapter IV). We denote by $ba(\Sigma)$ the Banach space of all finitely additive set functions on Σ endowed with the total variation norm. It is isometrically isomorphic to the norm dual of $B_0(\Sigma)$. Note also that the weak* topology $\sigma(ba, B_0)$ of $ba(\Sigma)$ coincides with the eventwise convergence topology. Throughout the paper, we assume that any subset of $ba(\Sigma)$ is endowed with the topology inherited from the weak* topology.

Given a mapping $u : X \to \mathbb{R}$, the function $u(f) : S \to \mathbb{R}$ is defined by $u(f)(s) = u(f(s)), \text{for all } s \in S$. We note that $u(f) \in B_0(S, \Sigma)$ whenever f belongs to $\mathcal{F}$. If x belongs to X, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$.

Additionally, we assume that the set of consequences X is a convex subset of a vector space. For instance, this is the case if X is the set of all simple lotteries on a set of *outcomes* Z. In fact, it is the classic setting of Anscombe and Aumann (1963) as presented by Fishburn (1970).

Using the linear structure of X, we can define as usual for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$ the act $\alpha f + (1-\alpha)g : S \to X$ by $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s)$, $\forall s \in S$. Also, given two acts $f, g \in \mathcal{F}$ and an event $E \in \Sigma$, we denote by fEg the act delivering the consequences $f(s)$ in E and $g(s)$ in E^c.

We denote by $\Delta := \Delta(\Sigma)$ the set of all (finitely additive) probability measures $p : \Sigma \to [0, 1]$, endowed with the topology inherited from the weak* topology as discussed above. We say that a mapping $\eta : \Delta \to [0, \infty]$ is grounded if

$$\{\eta = 0\} := \{p \in \Delta : \eta(p) = 0\} \neq \emptyset.$$

The effective domain of $\eta : \Delta \to [0, \infty]$ is defined by

$$\{\eta < \infty\} := \{p \in \Delta : \eta(p) < +\infty\}$$

and $\overline{\{\eta < \infty\}}$ denotes its closure. Also, η is lower semicontinuous if $\{\eta \leq r\}$ is closed for each $r \geq 0$. Moreover, we denote by Δ^σ the set of all countably additive probabilities in Δ. In particular, given $q \in \Delta^\sigma$, we denote by $\Delta^\sigma(q)$ the set of all probabilities in Δ^σ that are absolutely continuous w.r.t. q, *i.e.*,

[5] We say that a finite-valued function $f : S \to X$ is Σ-measurable if, for all $x \in X$, the set $\{s \in S : f(s) = x\} \in \Sigma$.

$\Delta^\sigma(q) = \{p \in \Delta^\sigma : p \ll q\}$, where $p \ll q$ means that $\forall A \in \Sigma$, if $q(A) = 0$, then $p(A) = 0$.

Functions of the form $\eta : \Delta \to [0, \infty]$ will play a key role in the paper because it will capture the subjective degree of plausibility for a DM of each prior. We denote by $\mathcal{N}(\Delta)$ the class of those functions such that η is grounded, convex and lower semicontinuous. As discussed in the Introduction, a mapping η is called an ambiguity index, and its formal justification is given by Proposition 8.

3 Axioms

The decision maker's preferences are given by a binary relation $\succsim$ on $\mathcal{F}$, whose usual symmetric and asymmetric components are denoted by $\sim$ and $\succ$. Next, we describe the axioms describing the structure of the preference relation $\succsim$.

Axiom 1 *Reflexivity*: For all $f \in \mathcal{F}$, $f \succsim f$.

Axiom 2 *Unambiguous Transitivity*: Suppose $f \succsim g$. If $h(s) \succsim f(s)$ for all s, then $h \succsim g$. Also, if $g(s) \succsim h(s)$ for all s, then $f \succsim h$.

Axiom 3 *C-Completeness*: For all constant acts x and y, $x \succsim y$ or $y \succsim x$.

Axiom 4 *Archimedean Continuity*: For all $f, g, h \in \mathcal{F}$ the sets:

$\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\}$ are closed in $[0,1]$.

Axiom 5 *Dominance Independence*: For all $f, g, h_1, h_2 \in \mathcal{F}$, and all $\alpha \in (0,1)$,

$$\text{if } f \succsim g \text{ and } h_1 \succsim h_2 \text{ then } \alpha f + (1-\alpha) h_1 \succsim \alpha g + (1-\alpha) h_2.$$

Axiom 6 *Unboundedness*: There are $x, y \in X$ such that for each $\alpha \in (0,1)$, there exist $z, \widehat{z} \in X$ such that

$$\alpha z + (1-\alpha) y \succ x \succ y \succ \alpha \widehat{z} + (1-\alpha) x.$$

We follow the standard notion of weak preference, *i.e.*, given two acts f and g, the relation $f \succsim g$ means that f *is at least as good as* g. In this way, Axiom 1 requires no elaboration because it says that any act is at least as good as itself. On the other hand, we relax the usual completeness and transitivity conditions about preferences over uncertainty acts. Concerning transitivity, we only impose the weaker and mild condition called *Unambiguous Transitivity* as presented in Axiom 2: it means that transitivity holds for acts f, g, h if some pair of those acts are pointwise related by a state-by-state dominance. This condition is fundamental also for Nau (1992) and Lehrer and Teper (2011). Also, note that Axiom 2 implies the following version of monotonicity: For every $f, g \in \mathcal{F}$, if $f(s) \succsim (\succ)\, g(s)$ for any $s \in S$ then $f \succsim (\succ)\, g$.

Monotonicity is a state-independence condition for both the weak and strict sense of preference, saying that decision makers always prefer acts delivering state-wise better payoffs, regardless of the state where the better payoffs occur.

Axiom 3 means that the possibility of incompleteness is entirely due to uncertainty.[6] For instance, Cettolin and Riedl (2013) verify in an experiment that observed behavior is consistent with incomplete preferences in decision making under uncertainty. Note that Axioms 2 and 3 imply that the restriction of $\succsim$ to constant acts is a complete preorder. Archimedean (mixture) Continuity is a standard technical assumption in an Anscombe and Aumann setting.

Axiom 6 is a technical assumption that guarantees that there are arbitrarily good as well as bad consequences, which implies that for any utility index $u : X \to \mathbb{R}$ representing $\succsim$ over X, we must have $u(X) = \mathbb{R}$. We note that this is a stronger version of Axiom A.7 proposed by MMR (2006).

Dominance Independence (Axiom 5) says that whenever the DM declares $f \succsim g$ and $h_1 \succsim h_2$, then the DM should be not only able to compare any the acts $\alpha f + (1-\alpha) h_1$ and $\alpha g + (1-\alpha) h_2$, but the DM should also declare the former (weakly) better than the latter. This is an appealing condition saying that mixtures of preferred acts should be also a preferred act when compared to the same mixture of the dominated ones.

Since the implication stated in the Dominance Independence axiom, in general, goes only in one direction, we have that this property does not imply the usual Independence axiom. Actually, recall that Independence says that:

Independence: For every $f, g, h \in \mathcal{F}$, and every $\alpha \in (0,1)$,

$$f \succsim g \;\Leftrightarrow\; \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h.$$

On the other hand, clearly Axiom 5 implies the following weak version of Independence:

W-Independence: For every $f, g, h \in \mathcal{F}$, and every $\alpha \in (0,1)$,

$$f \succsim g \;\Rightarrow\; \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h.$$

Our main result characterizes preferences that, in general, do not satisfy the converse of *W-independence*. On the other hand, such a class of preferences satisfies the following stronger version of Archimedean Continuity:

S-Continuity: For all $e, f, g, h \in \mathcal{F}$, the set

$$\{\alpha \in [0,1] : \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) e\} \text{ is closed in } [0,1].$$

An interesting fact is that a reflexive and *transitive* preference relation that satisfies W-independence and S-continuity satisfies Independence.[7] As a consequence, if we add transitivity to the set of our axioms, then we obtain a preference relation that satisfies Independence. Also, as is easy to see, under Dominance Independence, Independence implies Transitivity (see Lemma 20).

On the other hand, in the absence of transitivity, the Independence axiom does not imply Dominance Independence. For instance, the model of incomplete

[6] We do not consider the case where the source of incompleteness is tastes. For recent results in this line of investigations see Ok, Ortoleva, and Riella (2012) and Galaabaatar and Karni (2013).

[7] See, for instance, Lemma 1 on p. 127 of Dubra, Maccheroni and Ok (2004).

preferences proposed by Leher and Teper (2011) obeys the classical Independence axiom but it does not satisfy Dominance Independence unless it reduces to Bewley's model.

4 The Main Representation and Properties

Next, we present our main result characterizing the representation that relies on Axioms A1–A6.

Theorem 1 *Let $\succsim$ be a preference relation on the set of Anscombe–Aumann acts $\mathcal{F}$. Then the following conditions are equivalent:*

(1) $\succsim$ satisfies assumptions A.1–A.6.

(2) There exists an affine utility index $u : X \to \mathbb{R}$, with $u(X) = \mathbb{R}$, and a function $\eta : \Delta \to [0, \infty]$ that belongs to $\mathcal{N}(\Delta)$ such that for all f and g in $\mathcal{F}$,

$$f \succsim g \Leftrightarrow \int u(f)dp + \eta(p) \geq \int u(g)dp, \forall p \in \Delta.$$

For each u there is a unique $\eta^* : \Delta \to [0, \infty]$ consistent with our representation given by

$$\eta^*(p) = \sup\left\{\int (u(g) - u(f))\, dp : f \succsim g\right\}, \ \forall p \in \Delta.$$

One interesting aspect of our main result is the fact that the ambiguity index η has the same properties as a cost function that characterizes MMR variational preferences. Unlike variational preferences, our model allows examples of incomplete and intransitive preferences. Also, both models satisfy *different forms* of Independence. Another important aspect is that in our model the ambiguity index is constructed directly as a supremum of expected-utility differences of pairs of comparable acts. We note that this representation of η^* based on the relation $\succsim$ suggests a way in which this theory can be tested in experiments. On the other hand, the MMR representation essentially relies on the Fenchel–Moreau theorem for the variational representation of preferences. Furthermore, our representation also has an explicitly variational formulation. Let $\succsim$ be a variational Bewley preference represented by u and η^* as in our Theorem 1. It is easy to see that for all acts $f, g \in \mathcal{F}$,

$$f \succsim g \Leftrightarrow \min_{p\in\Delta}\left\{\int (u(f) - u(g))\, dp + \eta^*(p)\right\} \geq 0.$$

This observation leads us to consider a variational preference of MMR, denoted by $\succsim^\#$, and represented also by u and η^*. Let x^* be a consequence such that $u(x^*) = 0$. Then $f \succsim g$ if, and only if, $h \succsim^\# x^*$, where h is the act satisfying the equation $u(h) = u(f) - u(g)$. Later, we will study how to derive a variational preference as a weak extension of our model. Also, we will see in our Theorem 15 that the variational formulation above suggests a characterization of our model as a preference that emerges from a given variational preference. To summarize, despite the behavioral differences between our model and variational preferences, there are some interesting connections in terms of representations, which motivates the following definition:

Definition 2 *A preference $\succsim$ on $\mathcal{F}$ is called a variational Bewley preference if it satisfies Axioms A.1–A.6.*

In the case of Bewley preferences, the representation asserts that for all plausible priors $p \in C$, the expected net utility $\int (u(f) - u(g))dp$ should be nonnegative. On the other hand, our representation has a natural interpretation as a *weighted unanimity rule*, with the function η reflecting the weight given to a prior and higher values of η corresponding to priors given less weight. Bewley's incomplete preferences can be identified precisely with the addition of transitivity or independence (Theorem 5), and a prior receives the weight either 0 if plausible or ∞ when discarded. In this way, for each conjecture that Nature imposes the prior p as the true model, we may interpret $\eta^* (p)$ as the maximum expected net loss accepted by the DM in the face of such a prior. The weighted unanimity rule then asserts that $f \succsim g$ if, and only if, for all priors $p \in \Delta$, the corresponding net expected utility cannot generate a sacrifice that exceeds the loss given by $\eta (p)$, *i.e.*,

$$\int (u(f) - u(g))dp \geq -\eta^* (p) .$$

Also, note that for any prior $p \in \{\eta = 0\}$, Bewley's rule applies.

An interesting aspect of our representation rule concerns the indifference relation $\sim$. Since $f \sim g$ means that $f \succsim g$ and $g \succsim f$, we have the following

$$f \sim g \Leftrightarrow \eta (p) \geq \left| \int u(f)\, dp - \int u(g)\, dp \right|, \forall\, p \in \Delta.$$

Hence, an indifference between two acts is equivalent to the fact that any prior p generates a gap between the expected utilities bounded by $\eta (p)$. In particular, priors with full plausibility should entail the same expected utility for both acts. While in Luce's (1956) 'coffee and sugar' example a subject may exhibit indifference between any two cups of coffee with just a grain's difference of sugar, here a subject may exhibit indifference between any two acts having expected utility gaps bounded by the corresponding subjective plausibility. In a sense, when experts disagree that two acts f and g are equally desirable, this does not matter when the size of the disagreement is bounded by the degree of confidence in such experts. In this way, in our model there is a much wider class of indifference than Bewley's.

Following our Theorem 1, variational Bewley preferences can be represented by a pair (u, η^*). Hence, we will write u and η^* to denote our class of preferences. From now on, when we consider a variational Bewley preference, we will write u and η^* to denote the elements of such a pair. Next we give the uniqueness properties of this representation.

Corollary 3 *Two pairs* (u, η^*) *and* (u_1, η_1^*) *represent the same variational Bewley preference* $\succsim$ *if and only if there exist* $\alpha > 0$ *and* $\beta \in \mathbb{R}$ *such that* $u_1 = \alpha u + \beta$ *and* $\eta_1^* = \alpha \eta^*$*.*

In our main result (Theorem 1), the set Δ of all finitely additive probabilities plays a fundamental rule. Due to its very convenient analytical properties in applications, it is very useful to consider the case of countably additive probabilities. As we will study later, this leads to the construction of some interesting examples. Fortunately, the monotone continuity axiom proposed by Arrow (1970) ensures that, for our main result, only countably additive probabilities matter, provided Σ is a σ-algebra.[8] Formally, the monotone continuity axiom follows as:

Monotone Continuity: We say that a preference relation $\succsim$ on $\mathcal{F}$ is *monotone continuous* if for all consequences $x, y, z \in X$ such that $y \succ z$, and for all sequences of events $\{A_n\}_{n \geq 1}$ with $A_n \downarrow \emptyset$, there exists a $k \geq 1$ such that $y \succsim x A_k z$.

Proposition 4 *Let* $\succsim$ *be a variational Bewley preference represented by* (u, η^*)*. The following statements are equivalent:*

(i) The preference relation satisfies the monotone continuity axiom,

(ii) The set of all plausible priors $\{\eta^* < \infty\}$ *consists only of countably additive probabilities.*

4.1 Bewley Incomplete Preferences

We begin with the Knightian decision theory axiomatized by Bewley (2002). As we mentioned in the Introduction, the Bewley model is characterized by transitivity, an axiom that we have relaxed in our main result. In the next theorem, we show in detail the relation between transitivity and the decision rule obtained in Theorem 1. In particular, when we add transitivity, the only probabilities in Δ that matter are those to which the decision maker attributes maximum plausibility, that is, those in $\{\eta^* = 0\}$, other probabilities present null plausibility, *i.e.*, $\Delta = \{\eta^* = 0\} \cup \{\eta^* = \infty\}$. Also, transitivity implies that every probability that matters has the same degree of plausibility.

It is also interesting to note that there are other equivalent ways for getting the Bewley representation from a variational Bewley preference. First, the classical Independence is equivalent to transitivity under Dominance Independence, which means that any variational Bewley preference satisfying Independence can be represented by a Bewley unanimity rule.

Theorem 5 *Let* $\succsim$ *be a variational Bewley preference represented by* (u, η^*)*. The following are equivalent:*

(i) The preference $\succsim$ *satisfies Transitivity;*

(ii) The preference $\succsim$ *satisfies Independence;*

[8] See Proposition B.1 of GMM (2004) for the case of Bewley's preferences.

(iii) For all $f, g \in \mathcal{F}$

$$f \succsim g \text{ iff } \int u(f)dp \geq \int u(g)dp, \text{ for all } p \in \{\eta^* = 0\};$$

(iv) The function η^* takes only the values 0 and ∞.

Here, we would like to emphasize the role of assuming Unboundedness (Axiom 7) for uniqueness. Consider $u : X \to \mathbb{R}$ such that $u(X) = [N, M]$, a nonempty, convex and closed set of priors $C \subset \Delta$, and let $\succsim$ be the Bewley preference given by

$$f \succsim g \ \Leftrightarrow \int u(f)dp \geq \int u(g)dp, \text{ for any } p \in C.$$

Note that we can represent $\succsim$ using variational Bewley representations with many ambiguity indexes. Actually, both $\eta_1, \eta_2 : \Delta \to [0, \infty]$ where $\eta_1(p) = \eta_2(p) = 0$ for all $p \in C$, while otherwise

$$\eta_1(p) = +\infty \text{ and } \eta_2(p) = M - N,$$

is compatible with a variational Bewley representation of $\succsim$.[9] In this way, if there is no boundedness in our main result (Theorem 1) one can construct values that act *like* ∞ but are in fact finite, and thus the uniqueness wouldn't hold.

4.2 Complete Variational Bewley Preferences

The previous subsection showed that variational Bewley preferences satisfying the classical independence or transitivity are exactly given by the well known class of preferences proposed by Bewley (2002). On the other hand, a natural question arises if we ask what happens if we suppose that a variational Bewley preference $\succsim$ satisfies completeness.

In our representation, for f to be preferred to g, the "gap" between the EU of f and that of g according to each plausible probability model p must be large enough. Under completeness, the idea that the difference in (expected) utility between two alternatives should be large enough to trigger a strict preference is a natural model of intransitivity (*e.g.* the *skew-symmetric additive representation* of Fishburn (1989) in a Savage framework or, in a different setting, Ok and Masatlioglu (2007)). The next result shows that for the class of variational Bewley preferences, the completeness axiom contains transitivity, and thus leads to independence and a subjective expected utility representation.

Theorem 6 *If a variational Bewley preference $\succsim$ is complete, then it is transitive. In particular, $\succsim$ is a subjective expected utility preference.*

[9] Note that η_2 is the minimal ambiguity index in that representation.

4.3 A Comparative Notion

For the precise result concerning the characterization of η in the main result as an ambiguity index we need the following definition:

Definition 7 *We say that the preference relation* $\succsim_1$ *reveals more ambiguity than* $\succsim_2$ *if for any acts* f *and* g

$$f \succsim_1 g \Rightarrow f \succsim_2 g$$

The decision maker 2 (with utility index u_2 and ambiguity index η_2^*) has a richer unambiguous preference than the decision maker 1 (with utility index u_1 and ambiguity index η_1^*) because the decision maker 2 behaves as if he is better informed about the decision problem.[10]

Proposition 8 *The following statements are equivalent:*
a) The preference relation $\succsim_1$ *reveals more ambiguity than* $\succsim_2$
b) Both decision makers have the same attitude towards risk (w.l.o.g., $u_1 = u_2$*) and* $\eta_1^* \leq \eta_2^*$*.*

Now, assume that the subjective expected utility is the benchmark for the absence of ambiguity. We say that the preference relation $\succsim$ reveals ambiguity when it reveals more ambiguity than some subjective expected utility preference $\succsim_{SEU}$. As a consequence of Proposition 8 and since $\{\eta = 0\} \neq \emptyset$, the class of preferences characterized in Theorem 1 reveals ambiguity. We note that from our Theorem 6, completeness fully characterizes ambiguity neutrality. In the Appendix we study the class of divergence Bewley preferences with the help of this comparative notion.

4.4 Bewley Rationalization Rules

The main departure of variational Bewley preferences from the incomplete Bewley model is transitivity. There have been many critiques of the lack of transitivity, following the traditional viewpoint of rationality, and one normative way of dealing with this problem invokes the so-called *rationalization rules*, that transform each possibly intransitive preference relation into a transitive one. There are two natural possibilities in dealing with this kind of strategy for variational Bewley preferences. One is the *Bewley closure* of a given variational Bewley preference $\succsim$, defined as the smallest (transitive variational) Bewley preference $\overline{\succsim}$ that contains $\succsim$, *i.e.*, for any $f, g \in \mathcal{F}$

$$f \succsim g \Rightarrow f \overline{\succsim} g.$$

Another possibility is its dual concept, called the *Bewley interior*, namely, the largest (transitive variational) Bewley preference $\succsim^{\circ}$ that is contained in the original variational Bewley preference $\succsim$, *i.e.*, for any $f, g \in \mathcal{F}$

$$f \succsim^{\circ} g \Rightarrow f \succsim g.$$

[10] GMM (2004) used the same notion of comparison applied to the context of unambiguous (Bewley's) subrelations.

Next, we present the characterization of the Bewley closure and Bewley interior of a given variational Bewley preference.

Theorem 9 *Let* $\succsim$ *be a variational Bewley preference represented by* (u, η).
(a) *The Bewley closure of* $\succsim$ *is well defined and satisfies:*

$$f \overline{\succsim} g \;\Leftrightarrow \int u(f)\, dp \geq \int u(g)\, dp, \textit{ for all } p \in \{\eta = 0\}.$$

(b) *The Bewley interior of* $\succsim$ *is well defined and satisfies:*

$$f \succsim^{o} g \;\Leftrightarrow \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h, \; \forall \alpha \in [0,1], \; \forall h \in \mathcal{F}.$$

Moreover,

$$f \succsim^{o} g \;\Leftrightarrow \int u(f)\, dp \geq \int u(g)\, dp, \textit{ for all } p \in \overline{\{\eta < \infty\}}.$$

In the light of the experts interpretation, we may view the Bewley closure as the decision rule obtained after discarding all experts without a full degree of plausibility, which we can relate to Savage (1954), p. 58, "When our opinions, as reflected in real or envisaged action, are inconsistent, we sacrifice the unsure opinions to the sure ones".[11] On the other hand, we may view the Bewley interior as the decision rule that emerges after putting all plausible experts on the same level of relevance. As a consequence, in following the Bewley closure, the decision maker's preference becomes less incomplete, while following the Bewley interior makes the decision maker's preference more incomplete. Indeed, for any $f, g \in \mathcal{F}$, $f \succsim^{o} g \Rightarrow f \succsim g \Rightarrow f \overline{\succsim} g$ and the converse holds if, and only if, $\succsim$ is a Bewley preference.

An interesting aspect of the Bewley closure $\overline{\succsim}$ is that for a collection of acts $\{f_i\}_{i=1}^{n}$ that constitute a cycle for $\succsim$, *i.e.*,

$$f_1 \succsim f_2 \succsim \ldots \succsim f_{n-1} \succsim f_n \succsim f_1$$

the Bewley closure will declare full indifference

$$f_1 \overline{\sim} f_2 \overline{\sim} \ldots \overline{\sim} f_{n-1} \overline{\sim} f_n \overline{\sim} f_1.$$

Hence, in the perspective of the experts interpretation, since $\{\eta = 0\}$ captures the set of most reliable advisers, any cycle must be related to a complete agreement among those top experts.[12] As a consequence, any cycle should include

[11] Nishimura (2012) provided an interesting negative result concerning the transitive closure of some complete nontransitive preference relations $\succcurlyeq$ over an arbitrary set $A \times A$. He showed that for a variety of cases, such as semiorders (Luce, 1956), relative discounting time preferences (Ok and Masatlioglu, 2007), regret preferences (Loomes and Sugden, 1982; Bell, 1982), and collective preferences induced by a majority criterion, the corresponding transitive closure $\overline{\succcurlyeq}^{t}$ treat any pair of alternatives indifferently, *i.e.*, $\overline{\succcurlyeq}^{t} = A \times A$. In our case the transitive closure $\overline{\succsim}^{t}$ should satisfies $\overline{\succsim}^{t} \subset \overline{\succsim} \neq \mathcal{F} \times \mathcal{F}$. Hence, our previous result shows that no such negative result *a la* Nishimura (2012) holds for the class of variational Bewley preferences.

[12] Hence, a cycle has its origin in a disagreement between less reliable advisers, as discussed in the Introduction.

only acts in an indifference curve of the respective Bewley closure, and such indifferences sets are very small.[13] Actually, with two states of nature, indifference sets for incomplete Bewley preferences consist only of acts that return indifferent lotteries in both states. Since transitivity holds over consequences, we cannot have cycles in the two states case whenever $\{\eta = 0\}$ has more than one prior.

4.5 Some Illustrative Examples

In this subsection we start with the case of two states of nature, $S = \{s_1, s_2\}$, in order to further illustrate the model proposed in this paper.

For simplicity, let us consider the case of a risk neutral decision maker with a variational Bewley preference $\trianglerighteq$ over $\mathbb{R}^2$ (or, we may think in terms of utils). We assume an ambiguity index $\eta \in \mathcal{N}([0,1])$ and that $\trianglerighteq$ satisfies, for all $x, y \in \mathbb{R}^2$,

$$x \trianglerighteq y \Leftrightarrow \alpha x_1 + (1-\alpha) x_2 + \eta(\alpha) \geq \alpha y_1 + (1-\alpha) y_2 \text{ for all } \alpha \in [0,1].$$

Clearly, defining $\phi_{x,y} : [0,1] \to \mathbb{R}$ by

$$\phi_{x,y}(\alpha) := \alpha [(x_1 - y_1) + (y_2 - x_2)] + x_2 - y_2,$$

we obtain

$$x \trianglerighteq y \Leftrightarrow \phi_{x,y}(\alpha) \geq -\eta(\alpha) \text{ for all } \alpha \in [0,1].$$

Note that each pair x, y of acts generates a *line* determined by $\phi_{x,y}$ with $\phi_{x,y}(0) = x_2 - y_2$, $\phi_{x,y}(1) = x_1 - y_1$, and slope given by $(x_1 - y_1) + (y_2 - x_2)$. Next, we discuss some examples.

Example 10 *Consider the variational Bewley preference generated by the quadratic ambiguity index $\eta(\alpha) = (\alpha - 1/2)^2$. Let $y = (1,0)$ and $x = (0,100)$. In this case $x \succsim y$ does not hold because $x_1 - y_1 = -1 < -1/4 = -\eta(0)$. Even given the symmetry of the ambiguity index, it does not follows that y should be rejected in favor of x. Note that both opinions, 'state one will happen with probability one' and 'state two will happen with probability one' have the same degree of plausibility, namely, $1/4$. This means that the decision maker take seriously pieces of advice based on such opinions, and does not accept a loss higher than $1/4$ under the perspective of such opinions. Hence, the decision maker under consideration can not compare x and y. Note also that this preference is not transitive. Actually, consider $x = (1,0)$, $y = (1/4, 1/4)$ and $z = (0, 3/8)$. In this case it is easy to see that $x \trianglerighteq y$, $y \trianglerighteq z$ but not $x \trianglerighteq z$ (and also not $z \trianglerighteq x$).*[14]

[13] In a more precise way, any indifference set of a given Bewley preference is homeomorphic with an infinity intersection of hyperplanes, which has Lebesgue measure zero in the case of a finite number of states. Indeed, in the general case, an indifference set can be identify with the solutions of an infinity linear system in the form $\Psi_a = \{b \in B_0(\Sigma) : \int bdp = \int adp, \textit{ for all } p \in C\}$, for some $a \in B_0(\Sigma)$ and a nonempty, convex, and closed subset $C \subset \Delta(S)$.

[14] Note that both functions $\phi_{x,y}(\alpha) = \alpha - \frac{1}{4}$ and $\phi_{y,z}(\alpha) = \frac{3}{8}\alpha - \frac{1}{8}$ dominates η over $[0,1]$, but $\phi_{x,z}(0) = -\frac{3}{8} < -\frac{1}{4} = -\eta(0)$. In this example, the same is true for any $z = (0, z_2)$ with $z_2 \in (\frac{1}{4}, 1 - \frac{1}{\sqrt{2}}]$.

Example 11 *Let us consider the ambiguity index over* $[0,1]$ *given by* $\eta(\alpha) = -\ln\alpha$. *In this case, the decision maker gives full plausibility only to the opinion saying that state 1 is sure,* i.e., $p = (1,0)$. *On the other hand, the decision maker associates a maximal accepted loss,* $\eta(\alpha)$, *for all priors* $p = (\alpha, 1-\alpha)$ *saying that state one is possible but not sure,* i.e., $\alpha \in (0,1)$. *Actually, the decision rule is*

$$x \trianglerighteq y \Leftrightarrow \phi_{x,y}(\alpha) \geq \ln\alpha \text{ for all } \alpha \in [0,1].$$

By considering $x = (0,1)$, $y = (0,2)$, *and* $z = (0,3)$, *we obtain that* $x \doteq y$, $y \doteq z$ *but* $z \triangleright x$. *Hence, the symmetrical part* $\doteq$ *is not transitive. Note also that the decision maker cannot compare, for instance, the acts* $(20,0)$ *and* $(10,10)$.

Example 12 *Consider the variational Bewley preference generated by the* lower entropy *ambiguity index given by*

$$\eta(\alpha) = \min_{\beta \in \left[\frac{1}{3}, \frac{2}{3}\right]} \left\{ \alpha \ln\left(\frac{\alpha}{\beta}\right) + (1-\alpha)\ln\left(\frac{1-\alpha}{1-\beta}\right) \right\},\ 0 < \alpha < 1.$$

In this case, η *belongs to* $\mathcal{N}([0,1])$, *with* $\eta(0) = \eta(1) = \ln(3)$.[15] *Note that the Bewley closure of* $\trianglerighteq$ *is given by the Bewley preference* $\overline{\trianglerighteq}$ *where* $x \overline{\trianglerighteq} y \Leftrightarrow \phi_{x,y}(\alpha) \geq 0$ *for all* $\alpha \in \left[\frac{1}{3}, \frac{2}{3}\right]$. *In this case, we cannot have cycles for* $\trianglerighteq$ *because if so, the Bewley closure* $\overline{\trianglerighteq}$ *would declare that all acts involved in the cycle are equally desirable, but indifference sets for* $\overline{\trianglerighteq}$ *are singletons. Hence, the only possible violation of transitivity in this case involves cases in which* $x \trianglerighteq y$ *and* $y \trianglerighteq z$, *but the DM cannot compare* x *and* z.

Now, let us consider the case with more than two states of nature and compare our model with the regret theory of Bell (1982) and Loomes and Sugden (1982). Here, our point is that cycles can be viewed as a natural phenomenon in some circumstances of ambiguity combined with a regret reasoning.

Example 13 *Consider, for instance, the acts induced by a horse race, where there is a unique famous horse (#5) and four other unknown horses:*

Acts/states	s_1	s_2	s_3	s_4	s_5
f_1	**100**	90	80	**70**	100
f_2	90	80	**70**	**100**	100
f_3	80	**70**	**100**	90	100
f_4	**70**	**100**	90	80	100

Each bet pays the same amount of money in the state where the famous horse wins the horse race. But, concerning the unknown ones, in which we might assume the presence of ambiguity, a pessimistic regret reasoning similar to the previous example can lead to the cycle given by,[16] $f_4 \succ f_3 \succ f_2 \succ f_1 \succ f_4$. *In*

[15] See Lemma 2 in Strzalecki (2008) p. 31. Also, recall that $\lim_{\alpha \to 0^+} \alpha \ln\alpha = 0$.

[16] Loomes, Starmer and Sugden (1991) call a cycle in this direction an *unpredictable* cycle by the usual regret theory. Predicted cycles for regret theory are cycles in the opposite direction, which could be related to the notion of majority rule. In this way, Riella and Teper (2014) provided an axiomatic foundation that includes the class of majority rule preferences consistent with predicted cycles. Note that this class of preferences covers a special case of Lehrer and Teper (2011)'s model.

fact, this ranking is consistent with a variational Bewley preference represented by a utility index $u(x) = x$ *and an ambiguity index* η *where for all probabilities* $p = (\alpha_1, ..., \alpha_5)$,

$$\eta(p) = 10 \sum_{i=1}^{4} \alpha_i = 10 - 10\alpha_5,$$

note that $\eta(p) = 0$ *if, and only if,* $p = (0, 0, 0, 0, 1)$.

Finally, we note that by our Theorem 6, the intersection of our class of preferences and regret theory is just given by subjective expected utility theory because regret preferences are assumed to be complete.

5 Connections with Variational Preferences

In this section, we investigate the problem of extending (in a weak sense) a variational Bewley preference to a transitive and complete preference. This is motivated by the fact that on many occasions, the decision maker should be able to compare any pair of acts. First, we study an important way for obtaining a complete and transitive preference as a *weak caution extension* of a given variational Bewley preference.

Gilboa, Maccheroni, Marinacci and Schmeidler (2010) (GMMS, 2010) proposed a model where a decision maker is characterized by two preference relations capturing decisions that can be labeled in terms of rationality as objective or subjective, where the first is modeled through Bewley's unanimity rule and the second via the Gilboa and Schmeidler (1989)'s maxmin rule (see also, Chateauneuf (1991)), both with respect to the same set of multiple priors C. However, we follow the interpretation given by GMMS (2010) in their Theorem 4, where a DM starts with an incomplete preference relation $\succsim^*$ and a maxmin behavior emerges as a possible completion of $\succsim^*$ (see GMMS 2010, p. 763). This is consistent with the evidence in Cettolin and Riedl (2013) that DMs, rather than being ambiguity averse as proposed by Schmeidler (1989), are averse to making choices under uncertainty, which gives rise to indecisiveness. In this way, ambiguity aversion can be viewed as an extension property of incomplete preferences under uncertainty. While in GMMS, the DM starts with a Bewley preference, we assume in this Section that the DM starts with a variational Bewley preference.

In this way, our exercise is in spirit very different from the one of GMMS (2010), even though, formally, it is very close. Actually, from a conceptual point of view, the fact that starting with a variational Bewley preference as an objective criterion may not be ideal, since it violates transitivity unless we assume the Bewley's model (Theorem 5).

We denote by $\succsim^{**}$ the weak completion of $\succsim^*$, which we assume to be a monotone and continuous preorder (reflexive and transitive). In GMMS (2010), there are two key axioms about the interplay between an incomplete preference and its completion: Consistency and Default to Certainty.[17] Given two preference relations $\succsim^*$ and $\succsim^{**}$, for any $f, g \in \mathcal{F}$ and $x \in X$:

- *Consistency*: $f \succsim^* g$ implies $f \succsim^{**} g$.
- *Default to Certainty:* If not $f \succsim^* x$, then $x \succ^{**} f$.

In this paper we use a weak version of Consistency, given by

- *Weak Consistency:* $f \succsim^* x$ implies $f \succsim^{**} x$.

The next result obtains variational preferences of MMR (2006) as a *weak completion* of variational Bewley preferences by imposing conditions on the interplay of the two preferences $\succsim^*$ and $\succsim^{**}$. We note that under Weak Consistency and Default to Certainty, a variational representation of preferences can be derived without assuming the uncertainty aversion axiom of Schmeidler (1989) and Gilboa and Schmeidler (1989), and also without the weak certainty independence axiom of MMR (2006).

Theorem 14 *Let $\succsim^*$ be a variational Bewley preference represented by a pair (u, η^*) and suppose that $\succsim^{**}$ is a complete, monotone, and continuous preorder. If $\succsim^*$ and $\succsim^{**}$ jointly satisfy both Weak Consistency and Default to Certainty, then*

$$f \succsim^{**} g \;\Leftrightarrow\; \min_{p\in\Delta} \int u(f)\, dp + \eta^*(p) \geq \min_{p\in\Delta} \int u(g)\, dp + \eta^*(p).$$

It is worth noting that in GMMS (2010), their multiple priors version of this result, obtaining a pair of Bewley–Gilboa and Schmeidler representations, assumes Consistency.[18] Indeed, it is easy to see that for a pair of preferences satisfying our Theorem 14, if the ambiguity index is related to a multiple prior representation, then Consistency holds.[19] On the other hand, the assumption that Consistency holds for a pair of relations $\succsim^*$ and $\succsim^{**}$ as in our Theorem 14 constrains the original variational Bewley preference $\succsim^*$ to be a Bewley preference because the transitivity of $\succsim^{**}$ translates back to the original preference.[20]

Now, we provide an exercise in which we show how to retrieve a variational Bewley preference $\succsim^*$ via a given variational preference $\succsim^{\#}$. In GMMS (2010), this exercise is possible due the fact that, when $\succsim^{\#}$ has a Gilboa and Schmeidler's representation, the preference $\succsim^*$ coincides with the revealed unambiguous

[17] GMMS (2010) presents also a weak version of Default to Certainty called Caution. Indeed, in their model, both conditions are equivalent (see Faro and Lefort (2014), footnote 17, p. 9).

[18] See also Cerreia-Vioglio (2011).

[19] Indeed, the same proof of GMMS's result can be carried out assuming only weak consistency.

[20] Let $f, g, h \in \mathcal{F}$ s.t. $f \succsim^* g$ and $g \succsim^* h$. Hence, for all $p \in \Delta$, $\int (u(f) - u(h))\, dp + \eta(p) \geq \int (u(g) - u(h))\, dp$, and $\int (u(g) - u(h))\, dp + \eta(p) \geq 0$. Consistency gives

$$\min_{p\in\Delta} \int (u(f) - u(h))\, dp + \eta(p) \geq \min_{p\in\Delta} \int (u(g) - u(h))\, dp + \eta(p) \geq 0,$$

that is, $f \succsim^* h$.

preferences of $\succsim^{\#}$.[21] As discussed after our main result (Theorem 1), $\succsim^{*}$ can be represented alternatively as follows:

$$f \succsim^{*} g \Leftrightarrow \min_{p\in\Delta}\left\{\int\left(u\left(f\right)-u\left(g\right)\right)dp+\eta\left(p\right)\right\}\geq 0.$$

This variational formulation suggests that, given $f, g \in \mathcal{F}$, for a suitable choice of $h \in \mathcal{F}$ and $x \in X$ we have that $f \succsim^{*} g$ if, and only if, $h \succsim^{\#} x$. From this observation we derive the following result:

Theorem 15 *Assume that $\succsim^{\#}$ is a variational preference of MMR (2006) represented by V over $\mathcal{F}$ given by*

$$V\left(f\right)=\min_{p\in\Delta}\left\{\int u\left(f\right)dp+\eta\left(p\right)\right\},$$

with $u\left(X\right)=\mathbb{R}$, and $\eta \in \mathcal{N}\left(\Delta\right)$. Define $\succsim^{*}$ as follows

$$f \succsim^{*} g \Leftrightarrow \left\{\begin{array}{c} \forall h\in\mathcal{F},\ \forall x\in X\ \text{s.t.}\ \frac{1}{2}f\left(s\right)+\frac{1}{2}x\sim^{\#}\frac{1}{2}g\left(s\right)+\frac{1}{2}h\left(s\right),\forall s\in S,\\ h\succsim^{\#}x. \end{array}\right.$$

Then $\succsim^{*}$ is a variational Bewley preference represented by (u, η).

As an interesting illustration of the previous result, assume that $\succsim^{\#}$ is a variational preference with an entropic index $\eta = \theta R(\cdot \parallel q)$, $\theta \in (0,\infty)$.[22] This specification gives the well known Hansen and Sargent's (2001) robustness model. This class of preference has also a second-order expected utility representation (SOEU)[23] given by the functional (see, for instance, Strzalecki (2011) p. 56): for all $f \in \mathcal{F}$

$$V\left(f\right)=\phi_{\theta}^{-1}\left(\int\left(\phi_{\theta}\circ u\left(f\right)\right)dq\right),$$

where, $\phi_{\theta}\left(t\right) := -\exp\left(-\frac{t}{\theta}\right)$ for all $t \in \mathbb{R}$. Hence, by Theorem 15, the corresponding variational Bewley preferences $\succsim^{*}$ is a divergence Bewley preference (Appendix, 6.1) that satisfies the following decision rule: for all $f, g \in \mathcal{F}$

$$f \succsim^{*} g \Leftrightarrow \int \exp\left(-\frac{1}{\theta}\left[u\left(f\right)-u\left(g\right)\right]\right)dq \leq 1.$$

[21] The notion of revealed unambiguous preferences as proposed by GMM (2004) follows as: given a binary relation $\succeq$ over $\mathcal{F}$, the corresponding unambiguous preference $\succeq^{\Theta}$ is defined by: For all $f, g \in \mathcal{F}$, $f \succeq^{\Theta} g$ iff

$$\alpha f+\left(1-\alpha\right)h\succeq\alpha g+\left(1-\alpha\right)h,\ \forall h\in\mathcal{F},\forall\alpha\in\left(0,1\right).$$

GMM (2004) shows that if $\succeq$ is a MEU preference of Gilboa and Schmeidler represented by (u, C), then $\succeq^{\Theta}$ is a Bewley preference represented also by the pair (u, C).

[22] We might consider the case including the possibility of $\theta = \infty$, but in this case we obtain the SEU model with the reference prior q.

[23] Grant, Polak and Strzalecki (2009) provided an axiomatic foundation for the class of SOEU preferences.

Another interesting example follows by taking the class of monotone mean-variance preferences as in MMR (2006), and further explored by Maccheroni, Marinacci, Rustichini, and Taboga (2009). This is given by the subclass of variational preference with the Gini ambiguity index $\eta = \theta G(\cdot \parallel q)$ (Appendix 6.1). Assume that X is the set of all monetary lotteries and $u(t) = t$ as in MMR (2006, p. 1474). We note that in this case, by Theorem 15, the corresponding variational Bewley preferences $\succsim^*$ is also a divergence Bewley preference that satisfies: for all $f, g \in \mathcal{F}$

$$f \succsim^* g \Leftrightarrow \min_{p \in \Delta} \left\{ \int (f - g)\, dp + \theta G(p \| q) \right\} \geq 0,$$

and by Theorem 24 in MMR (2006), since the classic mean-variance preferences of Markowitz (1952) and Tobin (1958) coincide with monotone mean-variance preferences once they are restricted on their domain of monotonicity,[24] we get that if $f, g \in B_0(\Sigma)$ are s.t. $(f - \int f dq) - \left(g - \int gd\right) dq \leq \theta$, q-a.s. then

$$f \succsim^* g \Leftrightarrow \int f dq \geq \int g dq + \frac{1}{2\theta} Var_q(f - g),$$

where $Var_q(h) := \int \left(h - \int h dq\right)^2 dq$ denotes the variance of $h : S \to \mathbb{R}$.

6 Appendix

6.1 The Class of Divergence Bewley Preferences

In this part of the Appendix we study an important class of variational Bewley preferences. Assume there is an underlying probability measure $q \in \Delta^\sigma$. Given a convex continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi(1) = 0$ and $\lim_{t\to\infty} \phi(t)/t = 1$, the ϕ-divergence of $p \in \Delta$ with respect to q is given by

$$D_\phi(p \parallel q) = \begin{cases} \int \phi\left(\frac{dp}{dq}\right) dq, & \text{if } p \in \Delta^\sigma(q) \\ \infty, & \text{otherwise.} \end{cases}$$

The mappings $D_\phi(\cdot \parallel \cdot)$ are well known as standard divergences, a widely used notion of distance between distributions in statistics and information theory. The two most important divergences are the *relative entropy* given by $\phi(t) = t \ln t - t + 1$, and the *relative Gini concentration index* given by $\phi(t) = (t-1)^2/2$.

By Lemma 15 (p. 1463) in MMR (2006) we have that $D_\phi(\cdot \parallel q) \in \mathcal{N}(\Delta)$, which implies that any preference $\succsim$ on $\mathcal{F}$ that satisfy the following rule

$$f \succsim g \Leftrightarrow \int \{u(f) - u(g)\}\, dp \geq -\theta D_\phi(p \parallel q), \forall p \in \Delta,$$

[24] The domain of monotonicity in this case is given by

$$M = \left\{ f \in B_0(\Sigma) : f - \int f dq \leq \theta,\ q\text{-a.s.} \right\}.$$

where $\theta > 0$, and $u : X \to \mathbb{R}$ is an affine function, belong to the class of variational Bewley preferences. In view of their interesting properties, we call them *divergence Bewley preferences*. It is immediate that the next proposition follows from Proposition 4.

Proposition 16 *Divergence Bewley preferences are monotone continuous variational Bewley preferences with index of ambiguity level given by*

$$\eta^* : p \in \Delta \to \theta D_\phi (p \parallel q) .$$

Concerning the analysis of comparative attitudes, the next simple consequence of Proposition 8 shows that they depend only on the parameter θ, which can therefore be interpreted as a coefficient of ambiguity level. In order to be more specific about ϕ, we speak of ϕ-divergence Bewley preferences.

Corollary 17 *Given two ϕ-divergence Bewley preferences $\succsim_1$ and $\succsim_2$, the following statements are equivalent:*

a) The preference relation $\succsim_1$ reveals more ambiguity than $\succsim_2$.

b) Both decision makers have the same attitude towards risk (w.l.o.g., $u_1 = u_2$) and $\theta_1 \leq \theta_2$.

This result says that divergence Bewley preferences reveal more and more (respectively, less and less) ambiguity as the parameter becomes closer and closer to 0 (respectively, closer and closer to ∞). In fact, since for any $p \in \Delta^\sigma (q)$

$$\lim_{\theta \to \infty} \theta D_\phi (p \parallel q) = \begin{cases} \infty, \text{ if } p \neq q \\ 0, \text{ if } p = q \end{cases},$$

we obtain that divergence Bewley preferences tend, as $\theta \to \infty$, to rank acts according to the SEU criterion with subjective probability q. On the other hand, since for any $p \in \Delta^\sigma (q)$

$$\lim_{\theta \to 0} \theta D_\phi (p \parallel q) = 0,$$

we obtain that divergence Bewley preferences tend, as $\theta \to 0$, to rank acts according to the very cautious criterion. For example, when q has a finite support supp(q), such a cautious criterion says that[25]

$$f \succsim g \text{ iff } u(f(s)) \geq u(g(s)), \ \forall s \in \operatorname{supp}(q) .$$

The two most important divergences are the relative entropy and the relative Gini concentration index. When $\eta = \theta R(\cdot \parallel q) : \Delta \to [0, \infty]$, where $q \in \Delta^\sigma$ (σ-additive probability), and

$$R(p \parallel q) = \begin{cases} \int \log \left(\frac{dp}{dq} \right) dp \text{ if } p \ll q \\ \infty, \text{ otherwise} \end{cases}$$

[25] For the general case we need to assume some topological structure on the state space because

$$\operatorname{supp}(q) := \cap \{ E \subset S : E \text{ is closed and } q(E^c) = 0 \}$$

is the relative entropy index (w.r.t q). The other case is where $\eta = \theta G(\cdot \parallel q) : \Delta \to [0,\infty]$, where $q \in \Delta^\sigma$ and

$$G(p \parallel q) = \begin{cases} \frac{1}{2}\int \left(\frac{dp}{dq} - 1\right)^2 dq & \text{if } p \ll q \\ \infty, & \text{otherwise} \end{cases}$$

is the classic concentration Gini index.

6.2 Binary Relations on $B_0(\Sigma)$: Fundamental Lemmas

Given a binary relation $\trianglerighteq$ on $B_0(\Sigma)$, some fundamental properties for our theory follow.

- $\trianglerighteq$ is *Reflexive* if $a \trianglerighteq a$ for every $a \in B_0(\Sigma)$.
- $\trianglerighteq$ satisfies *Unambiguous Transitivity* if when given $a, b, c \in B_0(\Sigma)$ such that $a \trianglerighteq b$; if $c \geq a$, then $c \trianglerighteq b$, and if $b \geq c$, then $a \trianglerighteq c$.
- $\trianglerighteq$ is *Transitive* provided that whenever $a, b, c \in B_0(\Sigma)$, if $a \trianglerighteq b$ and $b \trianglerighteq c$, then $a \trianglerighteq c$.
- $\trianglerighteq$ is *Non-trivial* if there exist $a, b \in B_0(\Sigma)$ such that $a \triangleright b$, that is, $a \trianglerighteq b$ but not $b \trianglerighteq a$.
- $\trianglerighteq$ is *Continuous* if given any sequence $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ such that for all $n \geq 1$ $(a_n, b_n) \in \trianglerighteq$, if $a_n \overset{\|\cdot\|_\infty}{\to} a \in B_0(\Sigma)$ and $b_n \overset{\|\cdot\|_\infty}{\to} b \in B_0(\Sigma)$, then $(a, b) \in \trianglerighteq$. In other words, $\trianglerighteq$ is a closed subset of $B_0(\Sigma) \times B_0(\Sigma)$.
- $\trianglerighteq$ is *Archimedean Continuous* if for all $a, b, c \in B_0(\Sigma)$, the sets

  $$\{\alpha \in [0,1] : \alpha a + (1-\alpha)b \trianglerighteq c\}$$

  and

  $$\{\alpha \in [0,1] : c \trianglerighteq \alpha a + (1-\alpha)b\}$$

  are closed in $[0,1]$.
- $\trianglerighteq$ is *Affine* if for all $a, b, c \in B_0(\Sigma)$ and $\alpha \in (0,1)$, $a \trianglerighteq b$ if and only if $\alpha a + (1-\alpha)c \trianglerighteq \alpha b + (1-\alpha)c$.
- $\trianglerighteq$ is *Affine by Dominance* if for all $a, b, c_1, c_2 \in B_0(\Sigma)$ and $\alpha \in (0,1)$, if $a \trianglerighteq b$ and $c_1 \trianglerighteq c_2$, then $\alpha a + (1-\alpha)c_1 \trianglerighteq \alpha b + (1-\alpha)c_2$. It is worth noticing that this property says that $\trianglerighteq$ is a convex subset of $B_0(\Sigma) \times B_0(\Sigma)$.
- $\trianglerighteq$ is *Additive* when for all $a, b, c \in B_0(\Sigma)$, if $a \trianglerighteq b$, then $a + c \trianglerighteq b + c$.

Now, we present some useful results before the proofs of our main results.

Lemma 18 *Let $\trianglerighteq$ be a Reflexive, Unambiguous Transitive, and Affine by Dominance binary relation on $B_0(\Sigma)$. Then $\trianglerighteq$ is fully Monotone in the sense that,*[26] *for all $a, b \in B_0(\Sigma)$, $\forall s \in S$, $a(s) \geq (>)\, b(s) \Rightarrow a \trianglerighteq (\triangleright)\, b$.*

Proof. First, if $\forall s \in S$, $a(s) \geq b(s)$, then since $\trianglerighteq$ is Reflexive $b \trianglerighteq b$ and by Unambiguous transitivity, $a \trianglerighteq b$. Now, suppose that $\forall s \in S$, $a(s) > b(s)$. We already know that $a \trianglerighteq b$. Suppose that we also have that $b \trianglerighteq a$. Pick $c \in B_0(\Sigma)$ such that $\alpha b + (1-\alpha) c$ is a constant function. Since $\trianglerighteq$ is Affine by Domination, we have $\alpha b + (1-\alpha) c \trianglerighteq \alpha a + (1-\alpha) c$. Now, let a_* be the constant act that returns the worst consequence in $\alpha a + (1-\alpha) c$ in every state, *i.e.*, there exists s_* such that for all $s \in S$, $a_*(s) = (\alpha a + (1-\alpha) c)(s_*) = \min(\alpha a + (1-\alpha) c)$. By Unambiguous Transitivity, $\alpha b + (1-\alpha) c \trianglerighteq a_*$. On the other hand, since $\alpha b + (1-\alpha) c = \alpha b(s_*) + (1-\alpha) c(s_*)$ and $\alpha a(s_*) + (1-\alpha) c(s_*) = a_*$, by $a(s_*) > b(s_*)$ we must have

$$a_* = \alpha a(s_*) + (1-\alpha) c(s_*) > \alpha b(s_*) + (1-\alpha) c(s_*) = \alpha b + (1-\alpha) c,$$

a contradiction. ■

Lemma 19 *Let $\trianglerighteq$ be a Reflexive, Affine by Dominance, and Transitive binary relation on $B_0(\Sigma)$. Then $\trianglerighteq$ is Affine if and only if $\trianglerighteq$ is Additive.*

Proof. Suppose that $\trianglerighteq$ is Affine. Lemma 1 in GMMS(2010) says that Affinity is equivalent to: for all $a, b, c \in B_0(\Sigma)$ and for all $\lambda \geq 0$, if $a \trianglerighteq b$ then $\lambda a + c \trianglerighteq \lambda b + c$. Hence, in particular, $\trianglerighteq$ is Additive.

Conversely, suppose that $\trianglerighteq$ is Additive and let us use again Lemma 1 from GMMS (2010). Let $a, b, c \in B_0(\Sigma)$ and for all $\lambda \geq 0$, by our assumption if $a \trianglerighteq b$ then $a + c \trianglerighteq b + c$, which is the case for $\lambda = 1$. Suppose that $\lambda \in (0,1)$. By Additivity, $a \trianglerighteq b$ if and only if $a - b \trianglerighteq 0$, since $\trianglerighteq$ is Affine by Dominance and $0 \trianglerighteq 0$, we obtain $\lambda(a-b) \trianglerighteq 0$, that is, $\lambda a \trianglerighteq \lambda b$ and additivity implies that $\lambda a + c \trianglerighteq \lambda b + c$. Now, if $\lambda > 1$, using additivity by choosing $c_1 := (\lambda - 1) a$ we obtain that $\lambda a \trianglerighteq \lambda b + (\lambda-1)(a-b)$. Hence, if $\lambda \in (1,2]$, then $(\lambda-1)(a-b) \trianglerighteq 0$ and Additivity implies that $\lambda b + (\lambda-1)(a-b) \trianglerighteq \lambda b$, so by Transitivity, $\lambda a \trianglerighteq \lambda b$. Now, by induction, take $k \geq 3$ and suppose that if $\lambda \in (k-1, k]$, then $\lambda a \trianglerighteq \lambda b$. Now if $a \trianglerighteq b$ and $\lambda \in (k, k+1]$, then by our hypothesis for $\lambda \in (k-1, k]$ we have that $ka \trianglerighteq kb$. Hence, by Additivity and choosing $c_k := (\lambda - k) a$ we obtain that $\lambda a \trianglerighteq \lambda b + (\lambda - k)(a-b)$. So, since if $\lambda \in ((k, k+1]$ then $(\lambda-k)(a-b) \trianglerighteq 0$, Additivity implies that $\lambda b + (\lambda-k)(a-b) \trianglerighteq \lambda b$, so by Transitivity, $\lambda a \trianglerighteq \lambda b$. ■

Lemma 20 *Let $\trianglerighteq$ be a Reflexive and Affine by Dominance binary relation on $B_0(\Sigma)$. If $\trianglerighteq$ is Affine, then it is Transitive.*

Proof. Consider $a, b, c \in B_0(\Sigma)$ such that $a \trianglerighteq b$ and $b \trianglerighteq c$. Now, Affine by Dominance implies that $\frac{1}{2}a + \frac{1}{2}b \trianglerighteq \frac{1}{2}b + \frac{1}{2}c \equiv \frac{1}{2}c + \frac{1}{2}b$, and by Affinity, we obtain $a \trianglerighteq c$. ■

[26] Note that, $a(s) \trianglerighteq b(s) \Leftrightarrow a(s) \geq b(s)$ and $a(s) \triangleright b(s) \Leftrightarrow a(s) > b(s)$

Let us consider now the consequences of the Archimedean and continuity properties. The next lemma shows that, under Archimedean Continuity, Affine by Dominance implies Additivity. Before, we note that $\trianglerighteq$ on $B_0(\Sigma)$ is Additive if, and only if, for any $a, b, c, d \in B_0(\Sigma)$ such that $a - b = c - d$, $a \trianglerighteq b \Leftrightarrow c \trianglerighteq d$. Indeed, assume that $\trianglerighteq$ on $B_0(\Sigma)$ is Additive, then if $a, b, c, d \in B_0(\Sigma)$ are such that $a - b = c - d$ (*i.e.*, $a - c = b - d$) and $a \trianglerighteq b$, then $c + (b - d) = a \trianglerighteq b = d + (a - c)$ and by Additivity we obtain $c \trianglerighteq d$. For the converse, consider $a, b, c \in B_0(\Sigma)$ such that $a \trianglerighteq b$, hence since $a - b = (a + c) + (b - c)$, we get that $a + c \trianglerighteq b + c$.

Lemma 21 *An Archimedean Continuous and Affine by Dominance binary relation $\trianglerighteq$ on $B_0(\Sigma)$ is Additive.*

Proof. Consider $a, b, c, d \in B_0(\Sigma)$ such that $a \trianglerighteq b$. First, assume that $\lambda(a - b) = c - d$, for some $\lambda \in (0, 1)$. In this case, Affinity by Dominance implies that

$$c = \lambda a + (1 - \lambda) \frac{1}{1 - \lambda} (c - \lambda a) \trianglerighteq \lambda b + (1 - \lambda) \frac{1}{1 - \lambda} (c - \lambda a) = d.$$

Now, we assume the equality $a - b = c - d$. Note that Archimedean Continuity entails the ranking $c \trianglerighteq d$ if we show the inclusion

$$(0, 1) \subset \{\lambda \in [0, 1] : \lambda c + (1 - \alpha) d \trianglerighteq d\}.$$

Let $\lambda \in (0, 1)$, since $\lambda c + (1 - \alpha) d - d = \lambda(c - d) = \lambda(a - b)$, the previous case gives that $\lambda c + (1 - \alpha) d \trianglerighteq d$, and the proof is completed. ■

Next, we show a result giving conditions under which both notions of continuity are equivalent.

Lemma 22 *A Reflexive, Affine by Dominance and Unambiguously Transitive binary relation $\trianglerighteq$ on $B_0(\Sigma)$ is Continuous if and only if it is Archimedean Continuous.*

Proof. Clearly, if $\trianglerighteq$ is continuous, then it is Archimedean. Conversely, assume $\trianglerighteq$ is Archimedean. Let $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ $\|\cdot\|_\infty$-convergent sequences with limits $a, b \in B_0(\Sigma)$, respectively, such that for all natural number indices n, $a_n \trianglerighteq b_n$. Note that, by Lemma 21, $\trianglerighteq$ is additive. We need to show that $a \trianglerighteq b$, that is, by additivity, $a - b \trianglerighteq 0$, where 0 is the null constant function. Since for all $n \geq 1$, $a_n \trianglerighteq b_n$, the additivity of $\trianglerighteq$ allows us to write $c_n := a_n - b_n \trianglerighteq 0$ for all $n \geq 1$ and $c_n \stackrel{\|\cdot\|_\infty}{\rightarrow} a - b =: c$. Since $1_S \leq (\|c\|_\infty + 1) 1_S - c$, we note that for all $\varepsilon \in (0, 1)$ there exists $m \geq 1$ satisfying

$$c_m \leq c + \varepsilon 1_S \leq c + \varepsilon [(\|c\|_\infty + 1) 1_S - c].$$

Therefore, since

$$c + \varepsilon [(\|c\|_\infty + 1) 1_S - c] = \varepsilon [(\|c\|_\infty + 1) 1_S] + (1 - \varepsilon) c$$

for all $\varepsilon \in (0,1)$, there exists $m \geq 1$ such that $c_m \leq \varepsilon [(\|c\|_\infty + 1) 1_S] + (1-\varepsilon) c$, and since $c_m \trianglerighteq 0$, by Unambiguous Transitivity we obtain that for all $\varepsilon \in (0,1)$, $\varepsilon [(\|c\|_\infty + 1) 1_S] + (1-\varepsilon) c \trianglerighteq 0$. Hence,

$$(0,1) \subset \{\alpha \in [0,1] : \alpha [(\|c\|_\infty + 1) 1_S] + (1-\alpha) c \trianglerighteq 0\} =: A.$$

Furthermore, since $\trianglerighteq$ is Archimedean continuous, the inclusion above implies that $0 \in A = [0,1]$, *i.e.*, $a \trianglerighteq b$. ■

Lemma 23 *Let $\trianglerighteq$ be a Reflexive, Unambiguous Transitive, Continuous, and Affine by Dominance binary relation on $B_0(\Sigma)$. Then there exists a subjective expected utility preference $\trianglerighteq_q$ (with respect to a subjective probability $q \in \Delta$) such that for all $a, b \in B_0(\Sigma)$, if $a \trianglerighteq b$ then $a \trianglerighteq_q b$.*

Proof. First, we define an auxiliary function, which will be the key element for the proof of the main result in the next subsection. Consider the mapping $\eta^* : \Delta \to [0, +\infty]$ defined by,[27] for all $p \in \Delta$

$$\eta^*(p) = \sup_{(a,b) \in \trianglerighteq} \left(\int (b-a)\, dp \right).$$

Given a function pair $(a,b) \in \trianglerighteq$, we note that the mapping

$$\begin{aligned} \tau_{(a,b)} \quad &: \quad \Delta \to \mathbb{R} \\ p \quad &\mapsto \quad \tau_{(a,b)}(p) := \int (b-a)\, dp \end{aligned}$$

is linear and weak* continuous. Hence, by the usual arguments (See, for instance, Brézis (2010), pp. 10–11), the upper envelope

$$\eta^*(\cdot) = \sup_{(a,b) \in \trianglerighteq} \tau_{(a,b)}(\cdot)$$

is a weak* lower semicontinuous and convex function. Note also that since $\trianglerighteq$ is reflexive, $\eta^*(p) \geq 0$ for all $p \in \Delta$.

Note that if we show that there exists $q \in \Delta$ such that $\eta^*(q) = 0$, then we get the desired result because

$$\sup_{(a,b) \in \trianglerighteq} \left(\int (b-a)\, dp \right) = 0 \Rightarrow \forall (a,b) \in \trianglerighteq\,, \int a dp \geq \int b dp.$$

First, we note that $\inf_{p \in \Delta} \eta^*(p) = 0$. Indeed, since $\trianglerighteq$ is Affine by Dominance and Continuous, $\trianglerighteq$ is a closed[28] and convex subset of $B_0(\Sigma) \times B_0(\Sigma)$. Also, by the Banach–Alaoglu–Bourbaki theorem,[29] Δ is a weak* compact subset of

[27] Here, we follow the standard notation $(a,b) \in \trianglerighteq \Leftrightarrow a \trianglerighteq b$.

[28] With respect both the norm and the weak topologies on the product space $B_0(\Sigma) \times B_0(\Sigma)$. See, for instance, Brezis (2010) Theorem 3.7, p. 60.

[29] See, for instance, Brézis (2010), p. 66.

$ba(\Sigma)$. Moreover, it is easy to see that the function $p \mapsto \tau_{(a,b)}(p)$, is affine (hence concave) for each $p \in \Delta$. Another simple and important fact is that, by Lemma 18, if $a \trianglerighteq b$, then there exists $s_0 \in S$ such that $a(s_0) \geq b(s_0)$. Hence, by the Minimax Theorem[30]

$$\begin{aligned}\inf_{p\in\Delta} \sup_{(a,b)\in\trianglerighteq} \int (b-a)\, dp &= \sup_{(a,b)\in\trianglerighteq} \inf_{p\in\Delta} \int (b-a) = \\ & \sup_{(a,b)\in\trianglerighteq} \underbrace{\inf_{s\in S} (b-a)}_{\leq 0} \overset{((a,a)\in\trianglerighteq)}{=} 0.\end{aligned}$$

Finally, since Δ is a weak* compact subset of $ba(\Sigma)$ and η^* is weak* lower semicontinuous, by Baire's Theorem[31] there exists a $q \in \Delta$ with $\eta^*(q) = \inf_{p\in\Delta} \eta^* = 0$.[32] Hence, there exists $q \in \Delta$ such that if $a \trianglerighteq b$, then $a \trianglerighteq_q b$, where the binary relation $\trianglerighteq_q$ is the subjective expected utility preference induced by q. ■

In Ghirardato, Maccheroni and Marinacci (2004) we have the characterization of (reflexive) Bewley preferences.[33]

Lemma 24 *A binary relation $\trianglerighteq$ is a Non-trivial, Reflexive, Transitive, Continuous, Monotone, and Affine binary relation on $B_0(\Sigma)$ if and only if there exists a non-empty, weak* closed and convex subset C of Δ such that*

$$a \trianglerighteq b \Leftrightarrow \int adp \geq \int bdp \text{ for all } p \in C.$$

In this case, we say that $\trianglerighteq$ is a (Non-trivial) Bewley preference.

The next Lemma shows that if $\trianglerighteq$ is a Reflexive, Unambiguous Transitive, Continuous, and Affine by Dominance binary relation, then it is not an extremely cyclic binary relation, because its transitive closure is a proper subset of $B_0(\Sigma) \times B_0(\Sigma)$.

Lemma 25 *Let $\trianglerighteq$ be a Reflexive, Unambiguously Transitive, Continuous, and Affine by Dominance binary relation on $B_0(\Sigma)$. Consider a family of binary relations given by*

$$\Gamma(\trianglerighteq) := \left\{ \begin{array}{c} \trianglerighteq' : \trianglerighteq' \text{ is a Reflexive, Transitive, Continuous, Monotone,} \\ \text{and Affine by Dominance binary relation on } B_0(\Sigma) \text{ s.t. } \trianglerighteq \subset \trianglerighteq' \end{array} \right\}.$$

The Bewley closure *of $\trianglerighteq$ is defined by*

$$\trianglerighteq^* := \bigcap_{\trianglerighteq' \in \Gamma(\trianglerighteq)} \trianglerighteq' .$$

[30] See, for instance, Aubin and Ekeland (1984), chapter 6.

[31] See, for instance, Ok (2007), pp. 234, 237, Aliprantis and Border (2006) p. 44, or Brézis (2010) p. 11.

[32] In fact, we also obtain that the set of priors $\{\eta^* = 0\}$ is weak* closed and convex.

[33] Recall that Bewley (2002) characterizes the case where the primitive binary relation is irreflexive.

The binary relation $\trianglerighteq^*$ is a (Non-trivial) Bewley preference.[34]

Proof. First, we note that $\Gamma(\trianglerighteq)$ is nonempty because the trivial binary relation $\trianglerighteq^{\#} := B_0(\Sigma) \times B_0(\Sigma)$ belongs to $\Gamma(\trianglerighteq)$. Also, it is easy to see that $\trianglerighteq^*$ is Reflexive, Transitive, Continuous, Monotone, and Affine by Dominance. The fact that $\trianglerighteq^*$ is Non-trivial follows from Lemma 23, which guarantees the existence of a subjective expected utility preference $\trianglerighteq_q$ over $B_0(\Sigma)$ that belongs to $\Gamma(\trianglerighteq)$. Finally, that $\trianglerighteq^*$ is Affine follows from the same reasoning used in Lemma 1 of Dubra, Maccheroni and Ok (2004). Hence, by Lemma 24 we have that $\trianglerighteq$ is a Bewley preference. ■

Lemma 26 *Let $\trianglerighteq$ be a Reflexive, Unambiguous Transitive, Continuous, and Affine by Dominance binary relation on $B_0(\Sigma)$. Define $\trianglerighteq_*$ on $B_0(\Sigma)$ by*

$$a \trianglerighteq_* b \Leftrightarrow \lambda a + c \trianglerighteq \lambda b + c, \text{ for all } \lambda \geq 0 \text{ and } c \in B_0(\Sigma).$$

Then $\trianglerighteq_*$ is a Bewley preference. Also, if $\trianglerighteq_{\#} \subset \trianglerighteq$ is a Bewley preference, then $\trianglerighteq_{\#} \subset \trianglerighteq_*$. We call $\trianglerighteq_*$ the Bewley interior *of $\trianglerighteq$.*

Proof. First, we note that $\trianglerighteq_* \neq \emptyset$ because, for instance, $1 \trianglerighteq_* 0$ (note that $\lambda + c \trianglerighteq c$ for all $\lambda \geq 0$ and $c \in B_0(\Sigma)$). Indeed, it is straightforward to show that $\trianglerighteq_*$ is a Reflexive, Unambiguously Transitive, Continuous, and Affine by Dominance binary relation on $B_0(\Sigma)$. We note that $\trianglerighteq_*$ is Affine,[35] hence $\trianglerighteq_*$ is Transitive (Lemma 20) and by Lemma 24 we have that $\trianglerighteq_*$ is a Bewley preference.

Now, let $\trianglerighteq_{\#} \subset \trianglerighteq$ be a Bewley preference.[36] If for some $(a, b) \in \trianglerighteq_{\#}$ we have $(a, b) \notin \trianglerighteq_*$, then there exist $\lambda \geq 0$ and $c \in B_0(\Sigma)$ such that it is not true that $\lambda a + c \trianglerighteq \lambda b + c$. On the other hand, since $\trianglerighteq_{\#}$ is a Bewley preference and $a \trianglerighteq_{\#} b$, then $\lambda a + c \trianglerighteq_{\#} \lambda b + c$, a contradiction with $\trianglerighteq_{\#} \subset \trianglerighteq$. ■

Lemma 27 *If the binary relation $\trianglerighteq$ on $B_0(\Sigma)$ is Complete, Unambiguously Transitive, Continuous, and Affine by Dominance, then $\trianglerighteq$ is Transitive.*

Proof. Suppose $a, b, c \in B_0(\Sigma)$ satisfy $a \trianglerighteq b$ and $b \trianglerighteq c$. If *per contra* $a \trianglerighteq c$ fails to hold, then, since $\trianglerighteq$ is Complete, we have $c \triangleright a$. Now, consider the sequence $\{d_n\}_{n \in \mathbb{N}}$ where for each $n \geq 1$, $d_n := (1/n)\, 2\|a\|_\infty + (1 - (1/n))\, a$, clearly $d_n \overset{\|\cdot\|_\infty}{\to} a$. Hence, by Continuity and Completeness there exists $n_0 \in \mathbb{N}$ such that,[37] $c \triangleright (1/n_0)\, 2\|a\|_\infty + (1 - (1/n_0))\, a$. Also, note that by Unambiguous Transitivity, we must have $2\|a\|_\infty \trianglerighteq a$, and using Affine by Dominance, we

[34] Note that if $\trianglerighteq^{\#}$ is a Bewley preference such that $\trianglerighteq \subset \trianglerighteq^{\#}$, then $\trianglerighteq^* \subset \trianglerighteq^{\#}$ because $\trianglerighteq^{\#} \in \Gamma(\trianglerighteq)$.

[35] Given $a, b \in B_0(\Sigma)$, suppose that $a \trianglerighteq_* b$. Let $\alpha \geq 0$ and $d \in B_0(\Sigma)$, we need to show that $\alpha a + d \trianglerighteq_* \alpha b + d$. Let $\lambda \geq 0$ and $c \in B_0(\Sigma)$, since $a \trianglerighteq_* b$ we have $(\lambda\alpha)a + (\lambda d + c) \trianglerighteq (\lambda\alpha)b + (\lambda d + c)$, that is, $\lambda(\alpha a + d) + c \trianglerighteq \lambda(\alpha b + d) + c$, hence $\alpha a + d \trianglerighteq_* \alpha b + d$.

[36] For instance, take $\trianglerighteq'_{\#}$ defined by $a \trianglerighteq'_{\#} b$ if and only if $a(s) \geq b(s)$ for all $s \in S$.

[37] If not, by Completeness we would have $d_n \trianglerighteq c$ for all $n \geq 1$, and by Continuity, $a \trianglerighteq c$.

obtain that $(1/n_0)\, 2\,\|a\|_\infty + (1-(1/n_0))\, a \trianglerighteq a$. Now, let $\trianglerighteq_q$ be the subjective expected utility obtained in Lemma 23. Then,

$$c \trianglerighteq_q \frac{1}{n_0} 2\,\|a\|_\infty + \left(1-\frac{1}{n_0}\right) a \vartriangleright_q a \trianglerighteq_q b \trianglerighteq_q c,$$

a contradiction. ■

6.3 Proofs of the Results in the Main Text

Proof of Theorem 1:

Proof. (1) $\Rightarrow$ (2): First, we show that a representation *a la* von Neumann–Morgenstern (1944) holds for our preference when restricted to constant acts. Denote by $\succsim^c$ the restriction of $\succsim$ to a set of consequences X. Recall that C-Completeness imposes that $\succsim^c$ is complete. Also, by Unambiguous Transitivity, we have that $\succsim^c$ is transitive. Also, $\succsim^c$ satisfies Archimedean Continuity and Dominance Independence gives that for any $x, y, z \in X$ and for any $\alpha \in (0,1)$, if $x \succsim^c y$ then $\alpha x + (1-\alpha) z \succsim^c \alpha y + (1-\alpha) z$. By Dubra, Maccheroni and Ok (2004), the hypotheses of the mixture space theorem of Hernstein and Milnor (1953) are satisfied and there exists an affine function[38] $u : X \to \mathbb{R}$ such that $x \succsim y$ if and only if $u(x) \geq u(y)$. Moreover, u is unique up to a positive linear transformation. We note that Axiom 6 on Unboundedness implies that u is onto, *i.e.*, that $u(X) = \mathbb{R}$ (see, for instance, Lemma 29 of MMR 2006).

Clearly, given an affine and onto utility index u, we can define the mapping

$$\begin{array}{rcl} \Psi_u & : & \mathcal{F} \to B_0(\Sigma) \\ f & \mapsto & \Psi_u(f) := u(f), \end{array}$$

where $u(f) : S \to \mathbb{R}$ is defined by $u(f)(s) := u(f(s))$ for all $s \in S$, which is well defined. We note that Ψ_u is onto: if $a \in B_0(\Sigma)$, then there exist a finite partition $\{E_k\}_{k=1}^K$ and real numbers $\{r_k\}_{k=1}^K$ such that $a = \sum_{k=1}^K r_k 1_{E_k}$. Since u is onto, there exists a finite set of consequences $\{x_k\}_{k=1}^K$ such that $u(x_k) = r_k$ for all $k \in \{1, ..., K\}$. Now, take f such that for all $k \in \{1, ..., K\}$, $f(E_k) = \{x_k\}$, hence, given an index k, $u(f(s)) = r_k$ for all $s \in E_k$, which means that $\Psi_u(f) = a$. Finally, note that $u(f) = u(g)$ if and only if $u(f(s)) = u(g(s))\ \forall s \in S$ if and only if $f(s) \sim^c g(s)\ \forall s \in S$, and since $\succsim$ is monotone, $f \sim g$. Hence, Ψ_u is $\succsim$-injective, that is, if not $f \sim g$ then $\Psi_u(f) \neq \Psi_u(g)$. Given $a \in B_0(\Sigma)$, we denote by $\Psi_u^{-1}(a) := \{f \in \mathcal{F} : \Psi_u(f) = a\}$, hence if $f, g \in \Psi_u^{-1}(a)$, then $f \sim g$.

Now we define the binary relation $\trianglerighteq$ over the set $B_0(\Sigma) = \{u(f) : f \in \mathcal{F}\}$ by

$$a \trianglerighteq b \Leftrightarrow f \succsim g, \text{ for some } f, g \in \mathcal{F} \text{ such that } a = u(f) \text{ and } b = u(g).$$

[38] Recall that u is affine if for all $x, y \in X$ and $\alpha \in (0,1)$, $u(\alpha x + (1-\alpha) y) = \alpha u(x) + (1-\alpha) u(y)$,

We note that $\trianglerighteq$ is well defined on $B_0(\Sigma)$ and

$$a \trianglerighteq b \Leftrightarrow f \succsim g \text{ for all } (f,g) \in \Psi_u^{-1}(a) \times \Psi_u^{-1}(b).$$

By standard arguments, we have that $\trianglerighteq$ is **Reflexive**, **Non-trivial**, **Unambiguously Transitive**, **Archimedean Continuous**, and **Affine by Dominance**. So, by Lemmas 21 and 22, $\trianglerighteq$ is **Additive** and **Continuous**.

Now, the key step is to use the mapping $\eta^* : \Delta \to [0, +\infty]$ introduced in Lemma 23 and defined by, for all $p \in \Delta$

$$\eta^*(p) = \underset{(f,g)\in\succsim}{sup} \left(\int (u(g) - u(f))\, dp \right) = \underset{(a,b)\in\trianglerighteq}{sup} \left(\int (b-a)\, dp \right).$$

Since $\trianglerighteq$ is Additive, we note that $a \trianglerighteq b$ if and only if $a - b \trianglerighteq 0$, where 0 is the null constant function. Hence, putting $U_0 := \{c \in B_0(\Sigma) : c \trianglerighteq 0\}$, we can rewrite

$$\eta^*(p) = -\inf_{c\in U_0} \int cdp = \sup_{c\in U_0} \int -cdp.$$

Hence, as we saw before (Lemma 23), η^* is a non-negative, weak* lower semicontinuous and convex function such that $\{\eta^* = 0\} \neq \emptyset$.

Now, let us to show that we can use η^* to represent $\succsim$ as in the statement of the Theorem. If $f_0 \succsim g_0$, then $\eta^*(p) \geq \int (u(g_0) - u(f_0))\, dp$ for any $p \in \Delta$. Hence

$$\int u(f_0)dp + \eta^*(p) \geq \int u(g_0)dp \text{ for any } p \in \Delta.$$

Conversely, if $(f_0, g_0) \notin \succsim$, then $(a_0, b_0) \notin \trianglerighteq$, where $a_0 = u(f_0)$ and $b_0 = u(g_0)$. Furthermore, by defining $c_0 := a_0 - b_0$, we get that $c_0 \notin U_0$. Recall that U_0 is a nonempty, convex and closed subset of $B_0(\Sigma)$, and by the Separation Theorem[39] there exists $q \in ba(\Sigma)$ where

$$\int c_0 dq < \inf_{c\in U_0} \int cdq.$$

We note that for all $E \in \Sigma$, we have $q(E) \geq 0$. Indeed, if $q(F) < 0$ for some $F \in \Sigma$, since $n1_F \trianglerighteq 0$ for all $n \in \mathbb{N}$, we obtain that

$$\inf_{c\in U_0} \int cdq \leq \inf_{n\in\mathbb{N}} nq(F) = -\infty,$$

a contradiction because we saw that $\int c_0 dq \in \mathbb{R}$ is a lower bound for $\left\{ \int cdq : c \in U_0 \right\}$. Also, note that we can assume w.l.o.g. that $q(S) = 1$, that is, $q \in \Delta$, which implies that

$$\int c_0 dq < \inf_{c\in U_0} \int cdq = -\eta^*(q).$$

[39] See, for instance, Dunford and Schwartz (1958) p. 417, or Brezis (2010) p. 7.

Hence,

$$\int u(f_0)\,dq + \eta^*(q) < \int u(g_0)\,dq,$$

and gives its contrapositive: if for all $p \in \Delta$

$$\int u(f_0)\,dq + \eta^*(q) \geq \int u(g_0)\,dq,$$

then $f_0 \succsim g_0$.

(2) $\Rightarrow$ (1): This is straightforward.

Uniqueness: Given the variational Bewley preference $\succsim$ represented by (u, η^*), we know that $u(X) = \mathbb{R}$. Consider a variational preference $\succsim^{\#}$ represented also by the pair (u, η^*). Since $u(X)$ is unbounded, Proposition 6 in MMR (2006) entails that the mapping η^* is the unique function in $\mathcal{N}(\Delta)$ such that

$$f \succsim^{\#} g \Leftrightarrow \min_{p\in\Delta}\left\{\int u(f)\,dp + \eta^*(p)\right\} \geq \min_{p\in\Delta}\left\{\int u(g)\,dp + \eta^*(p)\right\}.$$

Now, by our Theorem 14 (this result does not rely on the uniqueness result) it is easy to see that if (u, η^{**}) also represents $\succsim$, then $\eta^{**} = \eta^*$. ■

Proof of Corollary 3:

Proof. Let (u, η^*) represent $\succsim$ as in Theorem 1. Taking another representation (u_1, η_1^*) of $\succsim$ as in Theorem 1, by its key equivalence, u and u_1 are affine representations of the restriction of $\succsim$ to the set of consequences X. Hence, by well known uniqueness results, there exists $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u_1 = \alpha u + \beta$. By the characterization of η^* obtained in Theorem 1 for any probability p, a simple calculation gives that $\eta_1^*(p) = \alpha\eta^*(p)$. ■

Lemma 28 *Consider a preference relation $\succsim$ as in Theorem 1 and some particular utility index $u : X \to \mathbb{R}$ consistent with $\succsim|_{X\times X}$. For any $f, g \in \mathcal{F}$ there exists a minimal $c_{(f,g)} \geq 0$ such that for any $c \geq c_{(f,g)}$*

$$f \succsim g \;\Leftrightarrow \int u(f)dp + \eta^*(p) \geq \int u(g)dp, \text{ for any } p \in \{\eta^* \leq c\}.$$

In fact, $c_{(f,g)} = \sup uog - \inf uof$.

Proof. The implication ($\Rightarrow$) is obvious. Now, suppose that $c \geq c_{(f,g)} := \sup u(g) - \inf u(f)$. Consider $p \in \Delta$ such that $\eta^*(p) \geq c_{(f,g)}$. Since $u(h) \in [\inf u(h), \sup u(h)]$ for $h \in \{f, g\}$, we have that $u(g) - u(f) \in [\inf u(g),\ \sup u(g)]$. Also,

$$\int (u(g) - u(f))\,dp \leq \|u(g) - u(f)\|_\infty \leq \sup uog - \inf uof \leq \eta^*(p).$$

Hence, if for some $c \geq c_{(f,g)}$

$$\int u(f)dp + \eta^*(p) \geq \int u(g)dp, \text{ for any } p \in \{\eta^* \leq c\},$$

then $f \succsim g$. ■

Proof of Proposition 4:

Proof. (i) implies (ii): Let $p \in ba(\Sigma)$ be an additive probability which is not countably additive. Then there exists a sequence of events $\{A_n\}_{n \geq 1}$ such that $A_n \downarrow \emptyset$ and $p(A_n) \downarrow \alpha > 0$. So, since $u(X) = \mathbb{R}$ for each $n \geq 1$, there exists some x_n such that $u(x_n) = n^{-1}$. Consider $z \in X$ such that $u(z) = 0$. Then monotonicity implies that $x_n \succ z$.

Now, by considering $x_m \in \left\{u^{-1}\left((\alpha n)^{-1} + m\right)\right\}$, $m \geq 1$, we obtain by the monotonic continuity axiom that there exist $k = k(n)$ such that

$$x_n \succsim x_m A_k z.$$

Hence,

$$\begin{aligned} \eta^*(p) &\geq \int (u(x_m A_k z) - u(x_n))\, dp \\ &= \left((\alpha n)^{-1} + m\right) p(A_k) - n^{-1} \\ &= mp(A_k) + \frac{1}{n}\left(\frac{p(A_k)}{\alpha} - 1\right), \end{aligned}$$

so, for any $m \geq 1$,

$$\begin{aligned} \eta^*(p) &\geq \lim_{n\to\infty}\left(mp(A_k) + \frac{1}{n}\left(\frac{p(A_k)}{\alpha} - 1\right)\right) \\ &= \lim_{n\to\infty} mp\left(A_{k(n)}\right) + \lim_{n\to\infty}\frac{1}{n}\left(\frac{p(A_k)}{\alpha} - 1\right) \\ &\geq m\alpha, \end{aligned}$$

which implies that $\eta^*(p) = \infty$. Hence, if $\eta^*(p) < \infty$, then $p \in ca(\Sigma)$.

(ii) implies (i): Let $x, y, z \in X$ be such that $y \succ z$ and suppose given a a sequence of events $\{A_n\}_{n\geq 1}$ with $A_n \downarrow \emptyset$. If $y \succsim x$, we have by monotonicity (y statewise dominates $xA_n z$) that $y \succsim xA_n z$ $\forall n \geq 1$. On the other hand, consider the case where $x \succ y$. We need to show that there exists some $n_0 \geq 1$ such that $y \succsim xA_{n_0}z$. By the previous Lemma, choosing $c = u(x) - u(y) + 1$ it is enough to show that for any $p \in \{\eta^* \leq c\}$, $u(y) + \eta^*(p) \geq \int u(xA_n z)\, dp$. Recalling that η^* is weak* lower semicontinous, we have that $\{\eta^* \leq c\}$ is a weak* compact set of countably additive probabilities, so it is a weak compact subset of countably additive probabilities. By Theorem IV.9.1 of Dunford and Schwartz (1958) it follows that if $\varepsilon > 0$ and $A_n \downarrow \emptyset$, then there exists n_o such that $p(A_n) < \varepsilon$ for any $n \geq n_0$ and all $p \in \{\eta^* \leq c\}$. Hence, putting $\varepsilon = [u(y) - u(z) + \eta^*(p)] / [u(x) - u(z)]$, we know that there exists n_0 such that

$$p(A_n) < [u(y) - u(z) + \eta^*(p)] / [u(x) - u(z)]$$

for any $n \geq n_0$ and for any $p \in \{\eta^* \leq u(x) - u(y)\}$. Hence, for any p such that $\eta^*(p) \leq c$,

$$u(y) + \eta^*(p) > p(A_n)\, u(x) + u(z)(1 - p(A_n)) = \int u(xA_n z)\, dp,$$

and we conclude that $y \succsim xA_n z$ for any $n \geq n_0$. ■

Proof of Theorem 5:

Proof. Assume that $\succsim$ is a variational Bewley preference represented by (u, η^*). Note that by our discussion on the section about the axioms, it is easy to see that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.[40] Also, Lemma 24 and the uniqueness result about η^* give that $(i) \Leftrightarrow (iv) \Leftrightarrow (v)$. ■

Proof of Theorem 6:

Proof. This result is an immediate consequence of Lemma 27. ■

Proof of Proposition 8

Proof. $a) \Rightarrow b)$ Concerning the same risk attitudes, it follows from GMM (2004), Corollary B.3, *i.e.*, we can take $u_1 = u_2 = u$. By assumption, $f \succsim_1 g \Rightarrow f \succsim_2 g$, *i.e.*, $\succsim_1 \subset \succsim_2$. Hence, the characterization of η^* gives that for all $p \in \Delta$, $\eta_1^* (p) \leq \eta_2^* (p)$.

$b) \Rightarrow a)$ Assume $f \succsim_1 g$, *i.e.*, $\int u(f)dp + \eta_1^*(p) \geq \int u(g)dp$, $\forall p \in \Delta$. Since $\eta_2^* \geq \eta_1^*$, we obtain that for any $p \in \Delta$, $\int u(f)dp + \eta_2^*(p) \geq \int u(f)dp + \eta_1^*(p) \geq \int u(g)dp$. ■

Proof of Theorem 9:

Proof. Let $\succsim$ be a variational Bewley preference represented by (u, η^*). By Lemma 25 and the comparative notion characterized in the Proposition 8, it is easy to see that the Bewley preference $\overline{\succsim}$ is represented by the same utility index u and the set of priors given by $\{\eta^* = 0\}$. Now, by Lemma 26 and again by the Proposition 8, it is easy to see that the Bewley preference $\succsim^\circ$ is represented by the same utility index u and the set of priors given by $\{\eta^* < \infty\}$. ■

Proof of Theorem 14

Proof. By our assumption, $\succsim^*$ is a variational Bewley preference represented by (u, η^*). We note that by weak consistency, $x \succsim^* y$ implies $x \succsim^{**} y$. By Default to Certainty, $x \succ^* y$ implies $x \succ^{**} y$. Therefore, $\succ^*$ and $\succ^{**}$ coincide on X, and the mapping $x \mapsto u(x)$ represents both preferences on X.

Let $x_f \in X$ be the certainty equivalent of f w.r.t. $\succ^{**}$. We note that by the continuity property of $\succsim^{**}$, for any $f \in \mathcal{F}$ there exists such a lottery x_f. Clearly, $f \succsim^{**} g$ iff $u(x_f) \geq u(x_g)$. So, for any act f, Default to Certainty yields $f \succsim^* x_f$. Hence,

$$\int u(f)\, dp + \eta^*(p) \geq u(x_f), \text{ for any } p \in \Delta,$$

therefore,

$$u(x_f) \leq \min_{p \in \Delta} \int u(f)\, dp + \eta^*(p).$$

If the strict inequality holds, there exists $y \in X$ such that

$$u(x_f) < u(y) < \min_{p \in \Delta} \int u(f)\, dp + \eta^*(p),$$

that is, $f \succsim^* y$ and $y \succ^{**} x_f$, by Weak Consistency $f \succsim^{**} y$ and $y \succ^{**} x_f$, and since $\succsim^{**}$ is a preorder we obtain $f \succ^{**} x_f$, which is impossible. ■

[40] See also Lemmas 19 and 20.

Proof of Theorem 15:

Proof. Our assumption says that there exists an affine function $u \in X^{\mathbb{R}}$ with $u(X) = \mathbb{R}$ and $\eta \in \mathcal{N}(\Delta)$ such that the functional over $\mathcal{F}$

$$V(f) = \min_{p \in \Delta} \left\{ \int u(f)\, dp + \eta(p) \right\}$$

represents $\succsim^{\#}$. By the definition of $\succsim^{*}$, given any $f, g \in \mathcal{F}$

$$f \succsim^{*} g \Leftrightarrow \left\{ \begin{array}{c} \forall h \in \mathcal{F},\, \forall x \in X \text{ s.t. } \frac{1}{2} f(s) + \frac{1}{2} x \sim^{\#} \frac{1}{2} g(s) + \frac{1}{2} h(s), \forall s \in S \\ h \succsim^{\#} x \end{array} \right. .$$

Since V as above represents $\succsim^{\#}$, it follows that

$$f \succsim^{*} g \Leftrightarrow \left\{ \begin{array}{c} \forall h \in \mathcal{F},\, \forall x \in X \text{ s.t. } u(h) = u(f) - u(g) + u(x) \\ \min_{p \in \Delta} \left\{ \int u(h)\, dp + \eta(p) \right\} \geq u(x) \end{array} \right. ,$$

that is, $f \succsim^{*} g \Leftrightarrow \min_{p \in \Delta} \left\{ \int [u(f) - u(g)]\, dp + \eta(p) \right\} \geq 0$. ■

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